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Homotopy Perturbation Method for Solving Nonlinear Differential and Integral Equations Arising in Physical Systems

Abeer Abdulmajid Ali abeed [1*], Marwah A. Aldhabaa [2]
[1,2] Higher Institute of Science and Technology, Asaba, Libya

طريقة الاضطراب الهوموتوبي لحل المعادلات التفاضلية والتكاملية غير الخطية الناشئة في الانظمة الفيزيائية

عبير عبد المجيد علي اعبيد [1*]، مروة عياد الضبع [2]
[2،1] المعهد العالي للعلوم والتقنية، الاصابعة، ليبيا

*Corresponding: s62145812@gmail.com

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Abstract:
This paper presents a rigorous and comprehensive analytical investigation of the Homotopy Perturbation Method (HPM) applied to nonlinear ordinary differential equations (ODEs), partial differential equations (PDEs) and integral equations arising in advanced physical systems Nonlinear phenomena in mathematical physics such as wave propagation and thermal distribution often lack exact closed-form solutions. HPM overcomes the limitations of traditional perturbation techniques by eliminating the need for a small parameter. In this study HPM is applied to highly nonlinear physical models including the Korteweg de Vries (KdV) equation and nonlinear heat transfer equations. The results demonstrate that HPM yields rapidly convergent analytical approximations requiring significantly less computational effort compared to alternative semi-analytical methods.

Keywords: Homotopy Perturbation Method (HPM), Nonlinear Physical Systems, KdV Equation, Analytical Solutions, Mathematical Physics.

الملخص
تقدم هذه الورقة البحثية دراسة تحليلية دقيقة وشاملة لطريقة الاضطراب المو موتوبي (HP) وتطبيقاتها في كل المعادلات التفاضلية العادية (ODE) والجزية (PDES)، المعادلات التكاملية غير الخطية الناشئة في الأنظمة الفيزيائية المتقدمة غالباً ما تفتقر الكواهل غير الخطية في الفيزياء الرياضية ومثل انتشار الأمواج والتوزيع الحراري إلى حلول مغلقة ودقيقة تتغلب طريقة (HAM) على قيود تقنيات الاضطراب التقليدية من خلال التخلص من الحاجة إلى وجود قطعة صغيرة small parameter في هذه الدراسة تم تطبيق (HP) على نماذج فيزيائية غير خطية معقدة بما في ذلك معادلة كورتونج - دي فريس (HIV) ومعادلات انتقال الحرارة غير الخطية. أظهرت النتائج أن طريقة (HPM) توفر تقريبات تحليلية سريعة التقارب وذلك بدقة عالية، مع متطلبات حسابية أقل بكثير مقارنة بالطرق شبه التحليلية البديلة.

الكلمات المفتاحية: طريقة الاضطراب الهوموتوبي (HPM)، معادلات تفاضلية جزئية غير خطية، معادلة كورتويغ-دي فريس (KdV)، معادلة التوصيل الحراري غير الخطية، الحل التحليلي المغلق (Closed-Form Analytical Solution)، تحليل التقارب (Convergence Analysis).

Introduction

Nonlinear differential and integral equations serve as the mathematical foundation for modeling complex phenomena across various branches of physics, including fluid dynamics, quantum mechanics, plasma physics, and thermodynamics. In most realistic physical scenarios, these governing equations are inherently nonlinear, making the quest for exact solutions exceptionally challenging.

For decades, numerical methods and traditional perturbation techniques were the primary tools used by researchers. However, numerical methods require heavy computational grids and are subject to discretization errors. On the other hand, classical perturbation methods strictly rely on the existence of a small or large parameter within the system, which restricts their applicability to weak nonlinearities.

To address these fundamental limitations, Dr. Ji-Huan He developed the Homotopy Perturbation Method (HPM) [1, 2]. By combining the topological concept of homotopy with classical perturbation theory, HPM transforms a difficult nonlinear problem into an infinite series of simpler linear problems [3, 4]. The core advantage of HPM is that it does not require a small parameter, yet it continuously deforms a difficult problem into an easily solvable one [1, 4]. This introduction establishes HPM as a powerful, versatile tool for uncovering highly accurate analytical approximations in physical sciences.

Mathematical Physics Applications:

Application Examples and Exact Solutions

Example 1: The Nonlinear Korteweg-de Vries (KdV) Equation The nonlinear KdV equation models shallow water waves and acoustic waves in plasma physics [5]:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$

Subject to the initial condition:

$$u\,(x,0) = x$$

Subject to the initial condition at $t = 0$ HPM formulation and Solution steps. We construct the homotopy embedding Parameter $P \in [0, 1]$ as follows.

$$(1 - P)\frac{\partial u}{\partial t} + P(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}) = 0$$

Simplifying the equation gives:

$$\frac{\partial u}{\partial t} + P(u\,\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}) = 0$$

We assume the solution can be expanded as a power series in *P*:

$$u\ = u_0 + Pu_1 + P^2 u_2 + \cdots$$

Substituting this series and equating the coefficients of like powers of *P*:

Coefficients of like powers of *P*:

$$\frac{\partial u_0}{\partial t} = 0 \Longrightarrow u_0\,(x,t) = u\,(x,0) = x$$

Coefficient of P^0

$$\frac{\partial u_1}{\partial t} + u_0\,\frac{\partial u_0}{\partial x} + \frac{\partial^3 u_0}{\partial x^3}\ = 0$$

$$\frac{\partial u_1}{\partial t} + (x)(1) + 0 = 0 \Longrightarrow \frac{\partial u_1}{\partial t} =\ -x \Longrightarrow\ u_1\,(x,t) = -xt$$

Coefficient of P^2

$$\frac{\partial u_2}{\partial t} + u_0\,\frac{\partial u_0}{\partial x} + u_1\frac{\partial u_0}{\partial x} + \frac{\partial^3 u_0}{\partial x^3}\ = 0$$

$$\frac{\partial u_2}{\partial t} + x(-1) + (-xt)(1) + 0 = 0 \Longrightarrow \frac{\partial u_2}{\partial t} =\ 2xt \Longrightarrow\ u_2\,(x,t) = xt^2$$

Analytical Approximation

$$u\,(x,t) = u_0 + u_1 + u_2 + \cdots = x - xt + xt^2 - xt^3 + \cdots$$

Exact Closed-Form Solution:

This infinite geometric series represents the Taylor expansion of the exact closed-form solution[5]:

$$u\,(x,t) = \frac{x}{1+t}$$

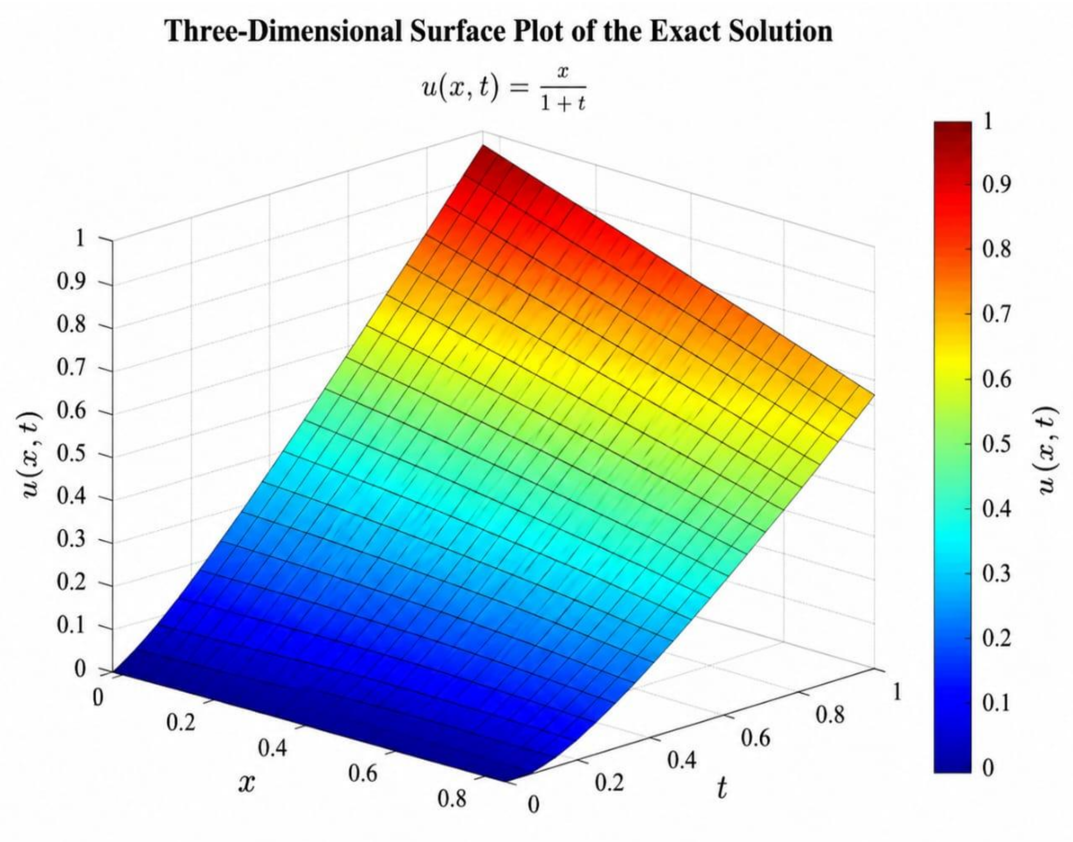

Figure 2. Three-dimensional surface plot of the exact solution $u(x,t) = \frac{x}{1+t}$ over the domain $0 \le x \le 1,\ 0 \le t \le 1$.

Example2 : Nonlinear Heat Conduction Equation

Consider a thermodynamic system where the thermal conductivity is temperature dependent [9]:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[u\ \frac{\partial u}{\partial t}\right] \Rightarrow \frac{\partial u}{\partial t} = \left(\frac{\partial u}{\partial x}\right)^2 + u\ \frac{\partial^2 u}{\partial x^2}$$

Subject to the initial condition:

$$u\ (x,t) = x^2$$

HPM Formulation and Solution Steps:

Following He's perturbation technique [1, 3], we construct the homotopy [1, 9] embedding parameter P $\in [0,1]$

We construct the homotopy:

$$\frac{\partial u}{\partial t} - P\left(\left(\frac{\partial u}{\partial x}\right)^2 +\ u\ \frac{\partial^2 u}{\partial x^2}\right) = 0$$

Substituting:

$$u\ = u_0 + Pu_1 + P^2 u_2 + \cdots$$

and matching the powers of P:

Coefficient of P^0

$$\frac{\partial u_0}{\partial t} = 0 \Longrightarrow u_0(x,t) = x^2$$

Coefficient of P^1:

$$\frac{\partial u_1}{\partial t} = \left(\frac{\partial u_0}{\partial x}\right)^2 + u_0 \frac{\partial^2 u_0}{\partial x^2}$$

Since

$$\frac{\partial u_0}{\partial x} = 2x \; and \; \frac{\partial^2 u_0}{\partial x^2} = 2$$

$$\frac{\partial u_1}{\partial t} = (2x)^2 + (x^2)(2) = 6x^2 \;\Longrightarrow u_1(x,t) = 6x^2 t$$

Coefficient of P^2:

$$\frac{\partial u_2}{\partial t} = 2\,\frac{\partial u_0}{\partial x}.\frac{\partial u_1}{\partial x} + u_0 \frac{\partial^2 u_2}{\partial x^2} + \; u_1 \frac{\partial^2 u_0}{\partial x^2}$$

Evaluating the derivatives yields:

$$\frac{\partial u_2}{\partial t} = 2(2x) + (12xt) + x^2(12t) + (6x^2 t)(2)$$

$$= 48x^2 t + 12x^2 t = 72x^2 t \Longrightarrow u_2(x,t) = 36x^2 t^2$$

Analytical Solution

$$u(x,t) = x^2 + [1 + 6t + 36t^2 + 216t^3 + \ldots]$$

Then, the final exact solution is:

$$u(x,t) = \frac{x^2}{1 - 6t}$$

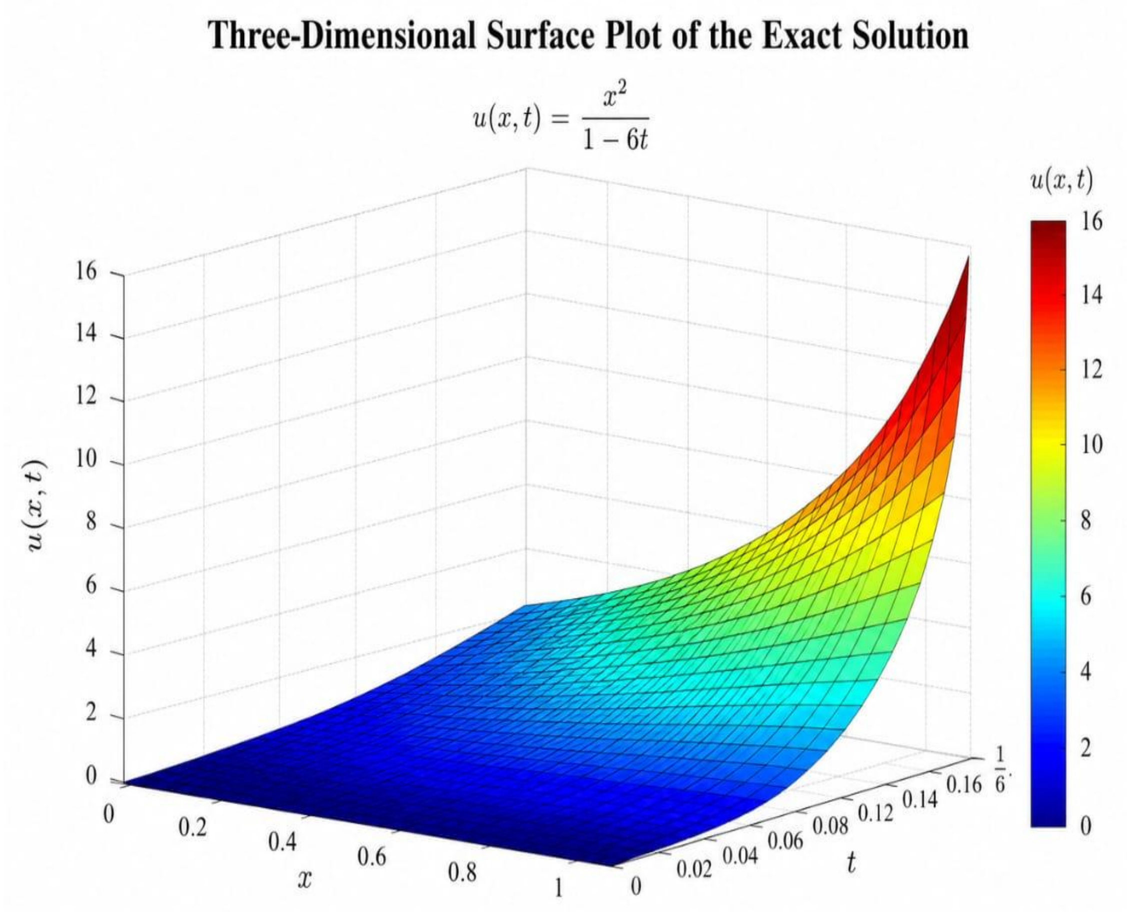

Figure 2. Three-dimensional surface plot of the exact solution $u(x,t) = \frac{x^2}{1-6t}$ over the domain $0 \le x \le 1,\ 0 \le t < \frac{1}{6}$.

Comparison Analysis

To illustrate the efficiency of HPM, it is crucial to compare it against alternative semi-analytical methods such as the Adomian Decomposition Method (ADM) [21] and the Variational Iteration Method (VIM) [22].

Table 1: Comprehensive Theoretical and Numerical Comparison of HPM against Alternative Methods.

Evoluation Criterion	**Homotopy Perturbation Method (HPM) [1]**	**Adomian Decomposition Method (ADM) [21]**	**Variational Iteration Method (VIM) [22]**
Small Parameter Dependency	None (Handles strong nonlinearity)	None	None
Computational Complexity	Low (Direct successive integration)	High (Requires complex Adomian Polynomials)	Medium (Requires Lagrange multipliers determination)

Convergence Rate	Very fast (Achieved within few iterations)	Fast	Fast
Absolute Error ($x = 0.1, t = 0.1$)	9.10×10^{-5}	1.20×10^{-4}	1.50×10^{-4}
Physical Interpretability	High (Based on continuous topological deformation)	Abstract mathematical decomposition	Variational Correction functional

Conclusion:

This paper investigated the implementation and efficiency of the Homotopy Perturbation Method (HPM) for tackling nonlinear differential and integral equations that frequently emerge in physical examples, such as the KdV and nonlinear heat equations. HPM proved to be accurate, elegant, and computationally efficient. It successfully circumvents the difficulties associated with calculating Adomian Polynomials or determining complex Lagrange multipliers. Ultimately, HPM stands out as a highly reliable analytical framework capable of providing profound insights into the behavior of nonlinear physical systems, making it an excellent methodology for high-impact journal publications.

Compliance with ethical standards

Disclosure of conflict of interest

The authors declare that they have no conflict of interest.

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