Scientific Journal for Publishing in Health Research and Technology
Volume 2, Issue 2, 2026, Pages: 295-305
Journal homepage: https://sjphrt.com.ly/index.php/sjphrt/en/index

On The Structure and Solvability of Pell and Pell-Type Equations: Algebraic and Statistical Perspectives

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حول البنية وقابلية الحل لمعادلة بيل ومعادلات بيل المعممة: دراسة جبرية وإحصائية

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Received: April 10, 2026 | Accepted: May 18, 2026 | Published: June 15, 2026

Abstract:
Pell's equation is one of the most classical Diophantine equations in number theory and is expressed in the form

$$x^2 - Dy^2 = 1$$

where D is a positive integer that is not a perfect square and x, y are integers. Despite its simple algebraic appearance, the equation possesses deep mathematical properties and rich structural connections with several branches of pure mathematics, including continued fractions, algebraic number theory, quadratic fields, and Diophantine analysis.

Historically, Pell's equation was studied extensively by ancient Indian mathematicians such as Brahmagupta and Bhāskara II, who developed sophisticated algorithms for solving quadratic Diophantine equations. Later, European mathematicians including Fermat, Euler, and Lagrange investigated the equation and established several fundamental theoretical results. In particular, Lagrange proved that Pell's equation always has infinitely many integer solutions when D is not a perfect square.

This paper presents a detailed study of Pell's equation, including its historical background, fundamental properties, classical solution methods using continued fractions, and important theorems describing the structure of its solutions. Furthermore, the paper explores several applications of Pell's equation in pure mathematics, particularly in algebraic number theory, quadratic forms, and Diophantine analysis. Worked examples are also provided to illustrate the methods used in solving Pell-type equations.

In addition, a statistical perspective is introduced to examine the distribution of solvable Pell-type equations for selected values of providing complementary insights to the classical theory and connecting number-theoretic behavior with probabilistic methods.

Keywords: Pell’s equation, Pell-Type equations, Diophantine equations,Solvability, Continued fractions, Quadratic fields, Algebraic number theory, Statistical analysis.

الملخص

تُعدّ معادلة بيل من أشهر المعادلات الديوفانتية في نظرية الأعداد، وتُكتب على الصورة:

$$x^2 - Dy^2 = 1$$

حيث D عدد صحيح موجب ليس مربعًا كاملًا، و x و y عددان صحيحان. على الرغم من بساطة مظهرها الجبري، إلا أن هذه المعادلة تتمتع بخصائص رياضية عميقة وروابط بنيوية غنية مع العديد من فروع الرياضيات البحتة، بما في ذلك الكسور المستمرة، ونظرية الأعداد الجبرية، والحقول التربيعية، والتحليل الديوفانتي.

تاريخيًا، دُرست معادلة بيل على نطاق واسع من قِبل علماء الرياضيات الهنود القدماء، مثل براهماغوبتا وبهاسكارا الثاني، الذين طوروا خوارزميات متطورة لحل المعادلات الديوفانتية التربيعية. لاحقًا، درس علماء الرياضيات الأوروبيون، بمن فيهم فيرما وأويلر ولاغرانج، هذه المعادلة، وأثبتوا العديد من النتائج النظرية الأساسية. على وجه الخصوص، أثبت لاغرانج أن معادلة بيل لها دائمًا عدد لا نهائي من الحلول الصحيحة عندما لا يكون D مربعًا كاملًا.

تقدم هذه الورقة دراسةً تفصيليةً لمعادلة بيل، تشمل خلفيتها التاريخية، وخصائصها الأساسية، وطرق حلها الكلاسيكية باستخدام الكسور المستمرة، ونظريات مهمة تصف بنية حلولها. علاوةً على ذلك، تستكشف الورقة تطبيقاتٍ متعددةً لمعادلة بيل في الرياضيات البحتة، لا سيما في نظرية الأعداد الجبرية، والصيغ التربيعية، والتحليل الديوفانتي. كما تُقدَّم أمثلة محلولة لتوضيح الطرق المستخدمة في حل معادلات من نوع بيل.

إضافةً إلى ذلك، يُقدَّم منظور إحصائي لدراسة توزيع معادلات بيل القابلة للحل لقيم مختارة من λ، مما يوفر رؤىً مكمِّلة للنظرية الكلاسيكية ويربط السلوك النظري للأعداد بالأساليب الاحتمالية.

الكلمات المفتاحية: معادلة بيل، معادلات من نوع بيل، معادلات ديوفانتية، قابلية الحل، الكسور المستمرة، الحقول التربيعية، نظرية الأعداد الجبرية، التحليل الإحصائي.

1. Introduction

Diophantine equations are polynomial equations for which integer solutions are sought. These equations are central to number theory because they reveal deep relationships between algebra and arithmetic properties of integers [1], [2], [4].

Among quadratic Diophantine equations, one of the most important and well-studied is **Pell’s equation**

$$x^2 - Dy^2 = 1$$

where D is a positive integer that is not a perfect square.

At first glance, the equation appears elementary; however, it exhibits remarkable mathematical properties. One of the most striking features is that the equation has infinitely many integer solutions. Moreover, these solutions possess a specific algebraic structure that connects the equation with quadratic number fields and units in algebraic integers [1], [5].

The study of Pell’s equation has a long history dating back to ancient Indian mathematics. Brahmagupta introduced an important identity for generating solutions, while Bhāskara II developed the Chakravala algorithm, which efficiently solves Pell-type equations. In Europe, Fermat studied these equations and challenged other mathematicians to find solutions [2], [9]. Later, Lagrange proved that the continued fraction expansion of square roots provides a systematic method for solving Pell’s equation [1], [6].

Because of these deep theoretical connections, Pell’s equation remains an important topic in modern number theory and algebra. Recent studies have continued to explore the properties and applications of Pell-type equations in modern number theory. In particular, their connections with algebraic number theory, quadratic forms, and computational number theory have attracted considerable attention. Several modern approaches investigate efficient algorithms for computing solutions and studying the algebraic structure of quadratic fields associated with Pell equations [10], [11].

Furthermore, this paper incorporates a statistical perspective to analyze the distribution of solvable Pell-type equations, drawing on basic concepts from probability and statistical analysis [12], [13].

2. Historical Background

The study of Pell's equation dates back to ancient mathematics. Early investigations of equations of this type appeared in the work of Brahmagupta in the seventh century. He introduced an important identity that allows the generation of new solutions from known ones, which became a fundamental tool in solving quadratic Diophantine equations [2], [9].

Later, Bhāskara II developed the Chakravala method, an efficient algorithm for finding integer solutions of equations of the form
$x^2 - Ny^2 = 1$. This method is considered one of the most advanced techniques of medieval mathematics [9].

During the seventeenth century, similar equations attracted the attention of European mathematicians. Pierre de Fermat studied several problems related to such equations and encouraged further research. The equation later became known as Pell’s equation due to a mistaken attribution by Leonhard Euler to John Pell [1].

A major theoretical development occurred when Joseph-Louis Lagrange proved that the continued fraction expansion of the square root of a non-square integer is periodic. This result established that Pell’s equation always has infinitely many integer solutions [1], [3].

3. Mathematical Formulation of Pell’s Equation

The standard form of Pell’s equation is

$$x^2 - Dy^2 = 1$$

Where:

- D is a positive integer
- D is not a perfect square
- x, y are integers

[3]

3.1 Definition

A pair of integers (x, y) satisfying the equation is called a **solution** of Pell's equation.

3.2 Definition

The smallest positive solution (x_1, y_1) is called the **fundamental solution**.
All other solutions can be generated from the fundamental solution [3].

4. Brahmagupta's Identity

A fundamental tool in the study of Pell equations is Brahmagupta's identity [2].

4.1 Lemma

If $x_1^2 - Dy_1^2 = k_1$ and $x_2^2 - Dy_2^2 = k_2$ then

$$(x_1x_2 + Dy_1y_2)^2 - D(x_1y_2 + x_2y_1)^2 = k_1k_2$$

Significance

This identity allows the construction of new solutions from known ones and forms the foundation of several classical solution methods [2], [9].

5. Infinite Solutions of Pell's Equation

5.1 Theorem

If D is a positive integer that is not a perfect square, then the equation

$$x^2 - Dy^2 = 1$$

has infinitely many integer solutions [1], [5].

Proof

Let (x_1, y_1) be the fundamental solution.
Consider $x_1 + y_1\sqrt{D}$
For any integer n, $(x_1 + y_1\sqrt{D})^n$ can be expressed as $x_n + y_n\sqrt{D}$

Multiplying by the conjugate gives

$$(x_n + y_n\sqrt{D})(x_n - y_n\sqrt{D}) = 1$$

Thus

$$x_n^2 - Dy_n^2 = 1$$

Hence infinitely many solutions exist [1]. ■

6. Continued Fractions Method

One of the most powerful methods for solving Pell's equation uses **continued fractions** [1], [3], [4].

A continued fraction has the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

The square root $\sqrt{D}$ has a **continued fraction expansion**

$$\sqrt{D} = [a_0; \overline{a_1, a_2, \dots, a_k}\,]$$

where the bar indicates that the sequence is periodic [3].

6.1 Theorem (Lagrange)

The continued fraction expansion of the square root of a non-square integer is periodic. Moreover, the convergents obtained from this expansion provide the fundamental solution of Pell's equation [1], [3].

Proof

Let D be a positive integer that is not a perfect square. The continued fraction expansion of $\sqrt{D}$ has the form

$$\sqrt{D} = [a_0; \overline{a_1, a_2, \dots, a_k}\,]$$

where the sequence eventually becomes periodic. The convergents

$$\frac{p_n}{q_n}$$

provide increasingly accurate rational approximations to $\sqrt{D}$.
For some value of n, the pair (p_n, q_n) satisfies

$$p_n^2 - Dq_n^2 = \pm 1$$

When the value equals 1, we obtain a solution of Pell's equation. The smallest such pair gives the fundamental solution [1]. ■

7. Chakravala Algorithm

The Chakravala algorithm is one of the most remarkable algorithms developed in ancient mathematics.
It is an iterative method used to solve equations of the form

$$x^2 - Dy^2 = k$$

The method repeatedly improves approximate solutions until the value $k = 1$ is obtained.

Steps of the Algorithm

1. Start with an initial triple (a, b, k)
2. Choose an integer m such that $a + bm$ is divisible by k
3. Minimize the value of $|m^2 - D|$

4. Generate a new triple

This process eventually produces the fundamental solution.

The Chakravala method is considered one of the most advanced algorithms of medieval mathematics [9].

8. Worked Examples

8.1 Example

Solve $x^2 - 2y^2 = 1$

Testing small values:

For $y = 2 \implies x^2 = 9$

Thus $x = 3$

Fundamental solution $(3,2)$

Other solutions:

$(17,12), (99,70), (577,408)$

8.2 Example

Solve $x^2 - 5y^2 = 1$

Try $y = 4 \implies x^2 = 81$

Thus $x = 9$

Fundamental solution $(9,4)$

Next solution:

$(9 + 4\sqrt{5})^2 = 161 + 72\sqrt{5}$

So $(161,72)$ is another solution.

These examples illustrate how Pell's equation possesses infinitely many solutions that can be generated systematically from the fundamental solution using algebraic identities or continued fraction expansions [3].

9. Pell-Type Equations

A more general form of Pell's equation is

$$x^2 - Dy^2 = N$$

where N is an integer.

These equations are called Pell-type equations.

Solving them often involves reducing the equation to the standard Pell equation.

These equations appear frequently in advanced number theory problems [5].

It is important to note that, unlike the classical Pell equation, the solvability of Pell-type equations is not guaranteed for all values of D and N. In many cases, the existence of integer solutions depends on arithmetic conditions such as congruences and divisibility properties. This makes Pell-type equations more complex and motivates further analytical and statistical investigation of their solution patterns [5], [7], [8].

In particular, equations of the form

$$x^2 - Dy^2 = \pm 2, \pm 4$$

are of special interest due to their irregular solvability [5], [8].

10. Applications in Pure Mathematics

10.1 Algebraic Number Theory

Pell's equation is closely related to **units in quadratic number fields** $\mathbb{Q}(\sqrt{D})$.
The solutions correspond to units in the ring $\mathbb{Z}[\sqrt{D}]$.
These units form an infinite cyclic group [5], [8], [11].

10.2 Quadratic Forms

Binary quadratic forms $ax^2 + bxy + cy^2$ are closely connected with Pell equations.
The classification of quadratic forms often leads to Pell-type equations [1], [10].

10.3 Approximation of Irrational Numbers

The solutions of Pell's equation give excellent rational approximations of irrational numbers such as $\sqrt{2}, \sqrt{3}, \sqrt{5}$.
This property plays an important role in number theory and numerical approximation [3].

10.4 Diophantine Analysis

Many Diophantine equations can be transformed into Pell equations.
Examples include certain exponential equations and equations involving squares.
Thus Pell's equation serves as a powerful tool in solving many integer equations [7], [8].

11. Pell's Equation and Units in Quadratic Fields

One of the most significant theoretical connections of Pell's equation arises in the study of quadratic number fields, which play a central role in Algebraic Number Theory. This connection provides a deeper algebraic interpretation of the structure of the solutions of Pell's equation.

Let D be a positive integer that is not a perfect square. Consider the quadratic number field

$$\mathbb{Q}(\sqrt{D})$$

which consists of all numbers of the form

$$a + b\sqrt{D}, \;\; a, b \in \mathbb{Q}.$$

Within this field, the set

$$\mathbb{Z}[\sqrt{D}] = \{a + b\sqrt{D} \,:\, a, b \in \mathbb{Z}\}$$

forms a ring. An element u in this ring is called a unit if it possesses a multiplicative inverse that also belongs to the same ring.

For an element

$$u = x + y\sqrt{D},$$

its norm is defined by

$$N(u) = \left(x + y\sqrt{D}\right)\left(x - y\sqrt{D}\right) = x^2 - Dy^2$$

If u is a unit in the ring $\mathbb{Z}\left[\sqrt{D}\right]$, then its norm must satisfy

$$N(u) = \pm 1.$$

Consequently, solving Pell's equation

$$x^2 - Dy^2 = 1$$

is equivalent to finding units in the quadratic ring $\mathbb{Z}\left[\sqrt{D}\right]$. In other words, every solution (x, y) of Pell's equation corresponds to a unit of the form $x + y\sqrt{D}$.

Furthermore, it can be shown that the set of units of this ring forms an infinite cyclic group generated by the fundamental solution of Pell's equation. This remarkable structure explains why Pell's equation always possesses infinitely many solutions when D is not a perfect square.

This relationship between Pell's equation and the algebraic structure of quadratic number fields represents one of the fundamental ideas in algebraic number theory and illustrates the deep connections between Diophantine equations and algebraic structures [5], [8], [11].

12. Statistical Observations on Pell-Type Equations

In addition to the theoretical analysis, we consider a statistical approach to study the solvability of Pell-type equations of the form

$$x^2 - Dy^2 = N$$

for selected values of D and $N = \pm 2, \pm 4$. Such an approach provides useful insights into the distribution of solutions and complements classical number-theoretic methods [12], [14].

12.1 Computational Data

We examine the solvability for $D = 2$ to 20:

D	***N***	**Solution Exists**
2	**2**	**Yes**
3	**2**	**No**
4	**2**	**No**
5	**4**	**Yes**
6	**2**	**No**
7	**-2**	**Yes**

8	**2**	**No**
9	**2**	**No**
10	**2**	**No**
11	**-2**	**Yes**
12	**2**	**No**
13	**4**	**No**
14	**2**	**No**
15	**-2**	**No**
16	**2**	**No**
17	**4**	**Yes**
18	**2**	**No**
19	**-2**	**Yes**
20	**4**	**No**

12.2 Statistical Analysis

Out of 19 tested cases, 6 admit integer solutions.

$$\text{Frequency} = \frac{6}{19} \approx 31.6\%$$

This simple frequency measure provides a quantitative description of solvability patterns [12], [13].

12.3 Observation

From the data, we observe that:

- Pell-type equations do not always admit integer solutions.
- Solvability depends strongly on arithmetic properties of D.
- The distribution of solvable cases is irregular.

These observations are consistent with the idea that such equations exhibit non-uniform behavior that may be studied using statistical tools [13], [14].

12.4 Interpretation

The results suggest that the solvability of Pell-type equations follows a non-uniform distribution. This behavior resembles probabilistic patterns and motivates further analytical and computational investigations [12], [14].

13. Discussion

The results presented in this study highlight the remarkable mathematical richness of Pell’s equation and its central position in number theory. Although the equation has a simple algebraic form, its solutions reveal deep structural properties that connect several areas of pure mathematics.

One of the most important observations is that Pell’s equation possesses infinitely many integer solutions whenever the parameter D is not a perfect square. This property reflects the underlying algebraic structure associated with quadratic number fields and explains why the equation plays a fundamental role in algebraic number theory. In particular, the correspondence between solutions of Pell’s equation and units in quadratic rings provides an elegant interpretation of the infinite sequence of solutions.

The methods discussed in this paper also illustrate the diversity of mathematical techniques used in solving Pell-type equations. Classical approaches such as continued fraction expansions provide an efficient and systematic method for determining the fundamental solution, while historical algorithms such as the Chakravala method demonstrate the remarkable achievements of early mathematicians in the study of Diophantine equations.

Furthermore, the applications presented in this study show that Pell's equation is not merely an isolated mathematical curiosity but rather a powerful tool that appears naturally in many areas of number theory. Its connections with quadratic forms, rational approximations of irrational numbers, and algebraic structures emphasize its broad mathematical significance.

Overall, the study of Pell's equation continues to provide valuable insights into the relationships between algebraic structures and Diophantine problems, making it an important topic of ongoing research in modern number theory.

The statistical observations provide an additional perspective that complements the classical algebraic theory and highlights irregular patterns in solvability, which can be interpreted using basic probabilistic concepts [12], [13].

14. Conclusion

This paper presented a comprehensive study of Pell's equation and its important role in number theory and pure mathematics. The study examined the historical development of the equation, beginning with the early contributions of ancient Indian mathematicians and continuing with the significant theoretical advancements made by European mathematicians.

The mathematical formulation of Pell's equation was discussed together with several fundamental theoretical results describing the structure of its solutions. In particular, it was shown that when the parameter D is not a perfect square, the equation possesses infinitely many integer solutions that can be generated systematically from the fundamental solution. Classical techniques such as continued fractions and the Chakravala algorithm were also presented as effective methods for determining these solutions.

In addition, the paper explored several important applications of Pell's equation in pure mathematics. These include its connections with quadratic forms, Diophantine analysis, and algebraic number theory. A particularly significant result is the relationship between Pell's equation and units in quadratic number fields, which provides a deeper algebraic interpretation of the equation and highlights its fundamental role in the theory of algebraic integers.

The examples presented in this study illustrate how the theoretical results can be applied in practice to determine solutions of Pell-type equations. These examples demonstrate the usefulness of the methods discussed and help clarify the structure of the solution sets.

Finally, the study confirms that Pell's equation remains one of the most elegant and influential equations in number theory. Its rich mathematical structure, historical importance, and wide range of applications ensure that it continues to attract attention in modern mathematical research.

Future work may focus on computational methods for solving generalized Pell equations, as well as further investigations of their connections with algebraic structures and modern developments in number theory.

Moreover, the statistical approach introduced in this study reveals irregular distribution patterns in the solvability of Pell-type equations and suggests directions for future research involving probabilistic and computational methods [12], [14].

Compliance with ethical standards

Disclosure of conflict of interest

The authors declare that they have no conflict of interest.

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