Graphs as Topological spaces: A Study of Block Topology and its Basic Properties

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الرسوم البيانية كفضاءات تبولوجية: دراسة في تبولوجيا الكتل وخصائصها الأساسية

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Abstract:

The interaction between graph theory and topology has yielded rich results for understanding many structural interpretations of discrete objects. This paper examines block spaces on undirected graphs. Fundamental concepts such as closure, density, and continuity were studied, and the conditions for connectedness were discussed, furthermore, the contractibility of block spaces and the fixed-point theory of continuous block-preserving functions were verified. Finally, the complete equivalence between topological homeomorphism and block-cut tree isomorphism were established.

Keywords: Block topological space, graph theory, connected space, continuity, contractible space, fixed- point

الملخص

لقد أدى التفاعل بين نظرية الرسوم البيانية (Graph Theory) والتبولوجيا إلى نتائج ثرية لفهم العديد من التفسيرات الهيكلية للكائنات المتقطعة. تبحث هذه الورقة في "فضاءات الكتل (Block spaces) "على الرسوم البيانية غير الموجهة. تمت دراسة مفاهيم أساسية مثل الانغلاق(Closure) ، والكثافة (Density)، والاستمرارية(Continuity) ، كما نوقشت شروط الترابط .(Connectedness) علاوة على ذلك، تم التحقق من قابلية الانكماش (Contractibility) لفضاءات الكتل ونظرية النقطة الثابتة للدوال المستمرة الحافظة للكتل. وأخيراً، تم إثبات التكافؤ التام بين التشاكل التبولوجي Topological homeomorphism)والتماثل لـ "شجرة قطع-كتل.(Block-cut tree isomorphism)" .

الكلمات المفتاحية :الفضاء التبولوجي الكتلي، نظرية الرسوم البيانية، الفضاء المترابط، الاستمرارية، فضاء قابل للانكماش، النقطة الثابتة.

1.Introduction

Many researchers have recently shown a clear interest in building a topological structure on graphs due to prospective applications not only in applied and pure mathematics but in many

fields, as in $[2], [4], [6], [7]$ and $[9]$. This connection allows the study of the graph from a topological perspective, opening up new research horizons in digital topology, networks, and other fields. In 2024, Macaso and Balingit defined the block topological spaces on undirected graphs G, which generated by its blocks (maximal 2-connected subgraphs without cut-vertices), and established some its basic attributes, but many foundational properties remain unexplored. In this paper, our motivation is making a more profound study of some properties of block topological space and the concepts associated with it. In section 2 outlines fundamental notions from graph theory and topology which are needed in our study. In addition, the definition of block topology under investigation is presented. Section 3 we examine several characteristics of block topology, including interior and closure points, dense subsets, and subspaces. We also study some standard results, such as the Alexandroff property of block topology. Section 4 is devoted to study the connectedness in this context. Lastly, section 5 of the paper focuses on continuity and establishes contractibility and the fixed-point theorem, furthermore discusses the homeomorphism classification.

2. Preliminaries

In this section, we give a brief of some basic concepts from graph theory and topology. Further details and background of these topics can be found in the sources [1,3,11,13].

2.1 Definition A graph $G = (V, E)$ consisting of a nonempty set of vertices V, and a set E of edges linking any pair of vertices of G. A graph G is a trivial graph if it has no any edges. G is directed (digraph) if each of whose edges is directed, and undirected otherwise.

2.2 Definition Let $G = (V, E)$ be a graph and $u, v \in V(G)$, then
- G is a simple graph, if $\forall u \in V$, $(u, u) \notin E$ and $\forall u, v \in V$, there is no multiple edges from one to another.
- Vertices u and v are endpoints of the edge $e \in E$ if $e = uv$.
- A vertex v is adjacent to vertex u if they are joined by an edge.
- A vertex u is isolated in a graph G if it is not an endpoint of any edge.

2.3 Definition Let $G = (V, E)$ be a graph and $u \in V(G)$, then
- A subgraph of G is a graph H such that $V(H) \subset V(G)$ and $E(H) \subset E(G)$.
- G is connected if there exists a path connecting each two of its vertices, and disconnected otherwise.
- A component of a graph G is a maximal connected subgraph.
- A cut-vertex u is a vertex whose removal increases the number of components.
- A block is a maximal connected subgraph without cut-vertex.
- An isolated vertex is the simplest block in G.
- Two blocks in a graph G share at most one vertex, which is cut-vertex.

2.4 Definition [1] A topology τ on a set X is a collection of subsets of X, which called open, such that the union of any collection of open sets is open, the intersection of a finite number of open sets is open, and both empty set and X are open. The pair (X, τ) is called a topological space.

2.5 Definition [1] Let (X, τ) be a topological space. Then
- The subspace topology τ_Y on the set $Y \subset X$ is $\tau_Y = \{U \cap Y : U \in \tau\}$.
- The class $\mathfrak{B}$ of open subsets of X is a base for a topology τ if each open set in τ is the union of members of $\mathfrak{B}$.
- A subbasis for a topology τ is a collection $\mathcal{S} \subset \tau$ of open subsets of X whose all-finite intersections form a base for τ on X.

In [10], the authors define the new topology on a simple undirected graph $G = (V, E)$ with blocks as follows;

2.6. Definition [10] Let $G = (V, E)$ be simple and undirected graph. The block topology, denote τ_B, is a topology that generated by the subbasis $\mathcal{S}_B$ such that:

$$\mathcal{S}_B = \{V(B_1), V(B_2), \dots, V(B_k)\}$$

Where $B_1, B_2, \dots, B_k$ are blocks or isolated vertices of G. The pair (V, τ_B) is called *block topological space,* simply *block space*, the elements of τ_B are called τ_B_open sets or simply open, which is defined as unions of finite intersections of elements of $\mathcal{S}_B$.

From here on, we suppose that all graphs in this paper are locally finite simple and undirected, and we call that the vertex that is neither cut-vertex nor an isolated vertex is pure vertex**.**

3. Basic notions for the theory of block topology

3.1. Definition Let $G = (V, E)$ be a graph, and let $v \in V$. The intersection of all blocks containing v is called *block of v* and it is denoted B_v , that is defined as follows:

$$B_v = \bigcap_i \{B_i : v \in V(B_i)\}$$

The family $\mathfrak{B}_B = \{V(B_v) : v \in V\}$ is a base for a block topology τ_B on V. In particular, it is the minimal base for the block topology τ_B.

From the last definition, it is easy to see that if v is a cut-vertex or isolated vertex then $B_v = \{v\}$, and if v is a pure vertex, then B_v is exactly equal to the block that contains v (it is only block which contains v).

3.2 Proposition Let u, v be pure vertices in a graph $G = (V, E)$**,** then $u \in V(B_v)$ if and only if $v \in V(B_u)$ Moreover, if $u \in V(B_v)$ then $B_u = B_v$ for every two pure vertices of V.

Proof. Let $v \in V$ and B_v be a block of v. Then $u \in V(B_v)$ if and only if u and v are connected by a path**,** if and only if $v \in V(B_u)$. For the second part, suppose that $u \in V(B_v)$ but $u \in V(B_u)$ thus $B_u \subseteq B_v$, in the same way we have $B_v \subseteq B_u$ equivalently $B_u = B_v$.   □

By considering blocks B_u of the graph $G = (V, E)$ as subgraphs of G. Block subspace topology of the block topology on these blocks B_u, which take two forms as follows;

3.3. Proposition Let $G = (V, E)$ be a graph and let B_u be a block of u , $u \in V$

 i. If u is a pure vertex, then the subspace topology on B_u is generated by the base $\{B_u, \{v_i\}, i \in I\}$ where v_i is a cut-vertex of G that contained in B_u.
ii. If B_u is a singleton (cut-vertex or isolated), then the corresponding subspace topology on B_u is an indiscrete topology.

3.4. Proposition Let $G = (V, E)$ be a graph, (V, τ_B) be a block space induced on G, and $H \subset G$ be a subgraph. Then $v \in V(H)$ is an *interior point* of H, if and only if $B_v \subseteq H$. The interior subgraph of H is defined as union of all blocks completely contained in H, and is denoted by $Int_B(H)$.

Proof. Suppose that $v \in V(H)$ is an interior point of H. then there is an open set $U \in \tau_B$ such that $v \in U \subseteq V(H)$, but $U = \bigcup_{u \in U} B_u$ this implies that $B_v \subseteq U \subseteq H$, and so $B_v \subseteq H$. Conversely, suppose that $B_v \subseteq H$. it follows from definition 3.1 that B_v is an open set, thus $v \in B_v \subseteq H$ and the proof is complete. □

3.5. Example Given a graph in the figure 1, the graph has the blocks $\{A, B\}, \{A, C\}$, and $\{C, D, E, F\}$. So, $B_A = \{A\}$, $B_B = \{A, B\}$, $B_C = \{C\}$ and $B_D = \{C, D, E, F\} = B_E = B_F$

$And \quad \tau_B = \{\emptyset, V, \{A\}, \{C\}, \{A, B\}, \{A, C\}, \{A, B, C\}, \{C, D, E, F\}, \{A, C, D, E, F\}\}$
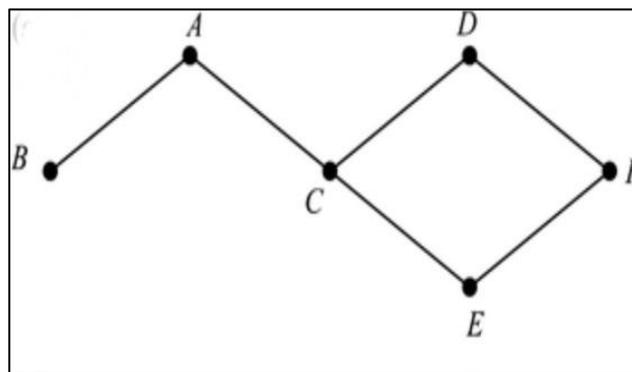

Figure. 1

Given the subgraph $H = G[\{A, D, F\}]$. Now, $B_A \subseteq H$, then A is an interior point of H. and since $B_B \nsubseteq H$, B is not an interior point of H. Similarly, note that the points C, D, E and F are not interior points of H. Consequently, $Int_B(H) = \{A\}$.

3.6 Remark A block B with two cut-vertices where $|V(B)| = 2$, is called a *bridge block*.
In the last example $\{A, C\}$ is a bridge block.

3.7 Corollary A subgraph $H \subseteq G$ is open in the block space if and only if it contains the blocks of every point within it. This means, H is open $\iff \forall v \in V(H), B_v \subseteq H$.

The following proposition determines when a particular point is a closure point in the block topological space;

3.8 Proposition Let $G = (V, E)$ be a graph, (V, τ_B) be a block space and $H \subseteq G$. Then $v \in V$ is a closure point of H if and only if $H \cap B_v \neq \emptyset$. The closure of subgraph H (is denoted by $cl_B(H)$) is union of all blocks that intersect H.

Proof. Let v be closure point of H. Then every open set U containing v intersects H, hence $U \cap H \neq \emptyset$. since $U = \bigcup_{u \in U} B_u$ and $v \in U$, this implies that $B_v \subseteq U$, therefore $B_v \cap H \neq \emptyset$. Conversely, suppose that $B_v \cap H \neq \emptyset$, so for every open set U contains v, it holds $\emptyset \neq B_v \cap H \subseteq U \cap H$, consequently, v is a closure point of H. □

3.9 Example Consider the graph G in Figure 1. take $H_1 = G[\{A, E\}]$, and $H_2 = G[\{C\}]$, then $cl_B(H_1) = \{A, B, D, E, F\}$, and $cl_B(H_2) = \{C, D, E, F\}$

3.10 Remark A cut-vertex (isolated vertex) belongs to $cl_B(H)$ if and only if it is already in H.

3.11 Proposition Let (V, τ_B) be a block space and $H_u = G[\{u\}]$. We have
i. If u is pure vertex, then $cl_B(H_u) = B_u$

ii. If u is cut-vertex, then $cl_B(H_u) = \bigcup\{B_v : u \in B_v\}$

iii. If u is isolated vertex, then $cl_B(H_u) = \{u\}$

Proof. i. $v \in cl_B(H_u)$ if and only if $\{u\} \cap V(B_v) \neq \emptyset$ if and only if $u \in V(B_v)$, from the proposition (3.3), it holds $cl_B(H_u) \subseteq B_u$. On the other hand, let $w \in V(B_u)$, so $w \in V(cl_B(H_u))$, because $u \in \{u\} \cap V(B_u)$ thus $B_u \subseteq cl_B(H_u)$.

ii. suppose that u is a cut-vertex, and there exists a path connecting u to w for some w in V. then $u \in V(B_w)$ and so $\{u\} \cap V(B_w) \neq \emptyset$, this means that $B_w \subseteq cl_B(H_u)$ which further implies the assertion.

iii. Let u be isolated vertex, then the only block contains u is $\{u\}$, so $cl_B(H_u) = \{u\}$.   □

3.12 Remark From the last propositions, note that if G has more than one cut-vertex then for each $u \in V$, $cl(B_u) \neq V$. In particular, $cl_B(H_u) \neq V$; $H_u = G[\{u\}]$.

3.13 Definition Let $(V(G), \tau_B)$ be a block space of the graph $G = (V, E)$, and $H \subseteq G$. the subgraph H is said to be *dense* in G if it contains at least one vertex of each block B_u for all $u \in V$ in the graph G. This means,

$$H \text{ is dense} \iff \{ H \cap B_u \neq \emptyset, \ \forall u \in V \}$$

3.14 Example Consider the graph which is shown in figure 2, note that the blocks of its vertices are $B_a = \{a, b, e\}, B_b = \{b\}, B_c = \{c\}, B_d = \{d, e\}, B_e = \{e\}$, and $B_f = \{b, f\}$, by taking all possible unions of those blocks produces the following block topology;

$$\tau_B = \left\{ \begin{array}{l} \emptyset, V, \{b\}, \{c\}, \{e\}, \{b, f\}, \{d, e\}, \{a, b, e\}, \{b, c\}, \{c, e\}, \{b, c, e, f\}, \{b, e\}, \\ \{b, d, e\}, \{b, c, e\}, \{b, c, f\}, \{b, e, f\}, \{b, d, e, f\}, \{c, d, e\}, \{a, b, d, e\}, \\ \{a, b, e, c\}, \{a, b, e, f\}, \{a, b, d, e\}, \{a, b, d, e, f\}, \{b, c, d, e\}, \{b, c, d, e, f\} \end{array} \right\}$$

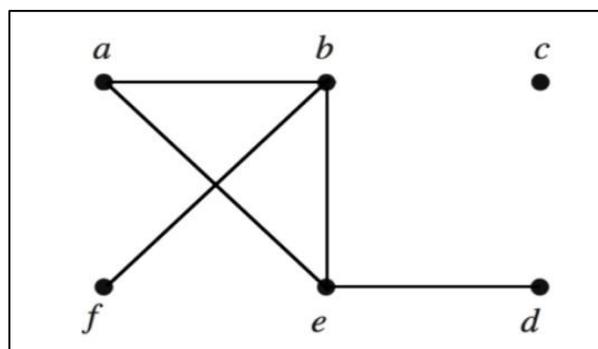So, the subgraphs $H_1 = G[\{c, b, e\}]$, and $H_2 = G[\{a, b, c, e\}]$ are dense in G.

Figure 2.

3.15 Corollary In connected graphs that contains no cut-vertices, each subgraph $H \subset G$ is dense.

3.16 Theorem Let $G = (V, E)$ be a graph, and let $M = \{v \in V : \{v\} \in \tau_B(G)\}$. Then G[M] is dense in G . In particular, If $H \subseteq G$, then H is dense if and only if $G[M] \subseteq H$.

Proof. Suppose that $M = \{v \in V : \{v\} \in \tau_B(G)\}$, this means that M contains all isolated vertices and cut-vertices of G, it is enough to prove that $G[M] \cap B_u \neq \emptyset$ for all $u \in V \setminus M$. since the blocks (that are not singletons) in G contain one cut-vertex, so each B_u intersects the subgraph G[M] , which means it is dense.

For the second statement, assume that H is dense and $v \in M$ if $v \notin V(H)$, then $\{v\} \cap H = \emptyset$ this implies that $H \cap B_v = \emptyset$ which is contradiction. Therefore $v \in V(H)$ and $G[M] \subseteq H$. On

the other hand, assume that $G[M] \subseteq H$, but from the first statement we have $G[M]$ is dense, therefore $G = cl_B (G[M]) \subseteq cl_B(H)$ and hence H is dense. □

3.17 Corollary Let $G = (V, E)$ be a tree, and $T = \{v \in V : d(v) > 1\}$. If $H \subseteq G$, then H is dense in G if and only if $G[T] \subseteq H$.

Proof. In a tree, each vertex v has $d(v) > 1$ is cut-vertex, so from the last theorem, the proof complete. □

3.18 Theorem Let $G = (V, E)$ be a graph. Then (V, τ_B) is Alexandroff space.

Proof. Let (V, τ_B) be a block topological space. Consider an arbitrary intersection of open sets $H = \bigcap_{i \in I} H_i$, such that each H_i is an open set in (V, τ_B). The aim is to prove that H is open. Since each open set H_i in τ_B is a block or union of blocks in the graph G. If $H = \emptyset$, then H is open and we done. If $H \neq \emptyset$, hence $V(H)$ equal to set of cut-vertices, which implies that $\bigcap_{i \in I}\{H_i\}$ is open. □

3.19 Theorem Let $G = (V, E)$ be a graph. Then the space (V, τ_B) is compact if and only if G is finite.

Proof. ($\Longrightarrow$) Let B_i is a block in the graph G for each i, if V is infinite, then the family $\{B_i\}_{i \in I}$ is an open cover of V which has no any finite subcover.

($\Longleftarrow$) observe.

4. Connectedness of block topology

Connectedness is a central concept in topology, describing the internal structure of space and the strength of connection between its parts. In [9], the authors proved that: a block topology induced by the graph G is connected, iff G is connected without cut-vertices. In this part, we study some conditions that affect the connectedness of block topology of a graph G;

4.1 Definition Let (V, τ_B) be block space induced by the graph $G = (V, E)$. A maximal connected subgraph $C \subseteq G$ is called a *block component* in (V, τ_B) if C is open, and if the inclusion $C \subset \acute{C}$ implies $\acute{C} \notin \tau_B$ or $\acute{C}$ is not connected in (V, τ_B), for any proper subgraph $\acute{C}$.

4.2 Remark Block components of topology τ_B constitute open subgraphs, and therefore they result from unions of open blocks, also they have similar behavior of components in topological spaces, namely that block components are pairwise disjoint and their union collectively cover the vertices set.

4.3 Proposition The block space (V, τ_B) induced by disconnected graph $G = (V, E)$ is disconnected.

Proof. Suppose that G be a disconnected graph with components $C_1, C_2, \ldots, C_n$ ($n \geq 2$), and τ_B is a block topology of G. Each block B_j of G is contained in exactly one component C_i, this means that $V(C_i) = \bigcup_{B_j \subseteq C_i} V(B_j)$, thus $V(C_i) \in \tau_B$. Now, define $U = V(C_1)$ and $W = \bigcup_{i=2}^n V(C_1)$, then U, W are nonempty open sets in $V(G)$ and $U \cup W = V(G)$. So $\{U, W\}$ is a separation and (V, τ_B) is disconnected. □

4.4 Corollary Let G be a disconnected graph. Then the number of block components in block space (V, τ_B) is equal to the number of connected components in G.

Proof. Suppose that a graph G is disconnected, and subgraphs $C_1, C_2, \ldots, C_n$ are connected components of G. The purpose is to show that there is a bijection between the connected components in G and block components in τ_B. For each i, $V(C_i) = \bigcup_{j \in J_i} V(B_j)$, B_j is a block which is open in (V, τ_B), therefore $C_i \in \tau_B$. Now, we claim that C_i is connected in $V(G)$. To prove the claim using the contradiction, suppose that C_i is disconnected. Then there exists open sets U and W that form a separation of C_i in $V(G)$ such that $V(C_i) = U \cup W$. Let $u \in U$

and $w \in W$ in C_i, but C_i is connected as a graph, so there exists a path connecting u and w. Hence, the vertices u and w belong to the same block or path P consists of a sequence of blocks connected via cut-vertices. This contradicts the assumption that $\{U, W\}$ is a separation. Thus C_i is connected. Finally, assume there is a connected open subgraph $Ć \supset C_i$, this means that $Ć$ contains vertices from more than one connected component in G, this contradicts the fact that the components C_i are disjoint, hence C_i is a maximal connected in τ_B. Consequently, the subgraphs $C_1, C_2, \dots, C_n$ are precisely the block components in (V, τ_B), and vice versa. So the graph G and the space (V, τ_B) have the same number of components. □

The following proposition provides the condition for a connected graph to generate a connected block space;

4.5 Proposition Let $G = (V, E)$ be a connected graph has at most one cut-vertex. Then a space (V, τ_B) is connected.
Proof. Suppose that G is connected and has at most one cut-vertex,
Case 1: G has no cut-vertices. From [9], the block space of G is indiscrete which is connected space, because the only open sets are $\emptyset$ and $V(G)$.
Case 2: G has exactly one cut-vertex. Let v be the unique cut-vertex. Then v must belong to each block in (V, τ_B). Thus, $V(G) = \bigcup_i B_i$ and $\bigcap_i B_i = \{v\} \neq \emptyset$ which mean that the space (V, τ_B) is connected.   □

4.6 Example The block space (V, τ_B) of the cycle graph C_n, the complete K_n, and the wheel W_n is the indiscrete, so it is connected.

4.7 Example Consider the graph $Ǵ = G - \{c\}$, where G is the graph that shown in Figure 2, then the block topology generated from $Ǵ$ is

$$\tau_B = \left\{ \begin{array}{c} \emptyset, \{b\}, \{e\}, \{b, f\}, \{d, e\}, \{a, b, e\}, \{b, e\}, \\ \{b, d, e\}, \{b, e, f\}, \{b, d, e, f\}, \{a, b, d, e\}, \\ \{a, b, e, f\}, \{a, b, d, e\}, \{a, b, d, e, f\} \end{array} \right\}$$

Which is connected, while the graph $Ǵ$ contains two cut-vertices. This implies that the converse of the previous proposition is not always true.

4.8 Proposition Let $G = (V, E)$ be a connected graph with bridge block. Then the block topology generated by G is disconnected.
Proof. Suppose that the connected graph G has a bridge block $B = \{u, v\}$, where u, v are cut-vertices. Then removing the uv edge separates the graph G into two connected components, C_u and C_v, such that
C_u is a component that contains of u and all vertices connected to u, except for v ; and
C_v is a component that contains of v and all vertices connected to v, except for u .
Now, let $U = V(C_u)$ and $W = V(C_v)$, then $\{U, W\}$ form a separation of $V(G)$, For by definition of the block components in block topology, the sets U and W are open in $V(G)$, are non-empty since they contain u, v respectively, and clearly, they are disjoint and their union cover $V(G)$. so the block topology of G is disconnected.   □

4.9 Example The graph G showing in Figure 2 has $\{A, C\}$ as a bridge block; consequently, the block topology of G is disconnected by a separation $\{\{A, B\}, \{C, D, E, F\}\}$.

It is worthy to note that, the blocks of the graph G are maximally connected subgraphs, therefore, the block topology has base of connected sets, and so is necessary locally connected.

5. Functions on block topological spaces

5.1 Definition [13] Let (X_1, τ_1) and (X_2, τ_2) be two topological spaces. A function $f : X_1 \longrightarrow X_2$ is said to be continuous if for each $A \in \tau_2$, $f^{-1}(A) \in \tau_1$. If the function f is bijective and, f and f^{-1} are continuous, we say that the spaces are homeomorphic and we write $X_1 \sim_h X_2$. In this case, the function f is called homeomorphism.

5.2 Proposition Let $f : V(G_1) \longrightarrow V(G_2)$ be a continuous function between two block spaces and c be cut-vertex in G_1. If c incident to the blocks $B_1, B_2, \dots, B_n$, then there exist a partition of the set $\{1.2, \dots, n\}$ into disjoint non-empty sets $I_1, I_2, \dots, I_m$ such that for each $j, k \in \{1,2, \dots, m\}$, the sets $\bigcup_{i \in I_j} f(B_i \setminus \{c\})$ and $\bigcup_{i \in I_k} f(B_i \setminus \{c\})$ must be contained in distinct connected components of $f(G_1) \setminus \{f(c)\}$

Proof. ($\Longrightarrow$) Assume f is continuous, and let $B_1, B_2, \dots, B_n$ be distinct blocks in G_1 that contain c. Consider two cases of behaver $f(c)$ inside $f(G_1)$:

Case 1. Suppose that $f(c)$ is not cut-vertex, then $f(G_1) \setminus \{f(c)\}$ is connected, therefore the connected subsets $f(B_1), \dots, f(B_n)$ must lie within the single component $f(G_1)$, so the condition of proposition are satisfied and then $m = 1$.

Case 2. $f(c)$ is cut-vertex in $f(G_1)$, then $f(G_1) \setminus \{f(c)\}$ is disconnected and contains the connected components $C_1, C_2, \dots, C_m$. For every B_i incident to c, since B_i is $2-$connected so $\{B_i \setminus \{c\}\}$ is connected. By continuity, the image $f(B_i \setminus \{c\})$ is connected, so it must be contained in exactly one the components $C_1, C_2, \dots, C_m$. Now, suppose, by contradiction, that there exists two blocks B_i and B_j such that $f(B_i \setminus \{c\})$ and $f(B_j \setminus \{c\})$ are contained in the same connected component of $f(G_1) \setminus \{f(c)\}$. Then there are a path P connecting a vertex in $f(B_i)$ to another vertex in $f(B_j)$. Now, by continuity of f, the preimage $f^{-1}(P)$ connects B_i and B_j without passing through c, this contradicts that c is cut-vertex separating them. Thus, we must have $m \geq 2$. This complete the proof. □

The following proposition shows that the continuous functions between block spaces preserve the fundamental block structure of graphs, as each block in the range space must come from some block in the domain space;

5.3 Proposition Let $f : V(G_1) \longrightarrow V(G_2)$ be a continuous function between two block spaces. If f is onto, then for every block B in G_2, there exists a block C in G_1 such that $f(C)$ is dense in B.

Proof. Let B be block in G_2. Since f is onto, the preimage $f^{-1}(B)$ is non-empty and it is open in $V(G_1)$ which is unions of blocks of G_1. Therefore, there exists at least one block of G_1 with $C \subseteq f^{-1}(B)$, by construction, $f(C) \subseteq B$. Now suppose for contradiction that $f(C)$ is not dense in B. By the definition 3.13 this means there is a sub-block $\acute{B} \subseteq B$ such that $f(C) \cap \acute{B} = \emptyset$ and $f(C) \subseteq B \setminus \acute{B}$, then $C \subseteq f^{-1}(B \setminus \acute{B})$. Since f is continuous and onto, $f^{-1}(\acute{B})$ is non-empty and open in $V(G_1)$. The sets $D_1 = f^{-1}(B \setminus \acute{B})$ and $D_2 = f^{-1}(\acute{B})$ are non-empty disjoint open sets whose union is $f^{-1}(B)$. However, $B = f(f^{-1}(B)) = f(D_1 \cup D_2) = f(D_1) \cup f(D_1)$ where $f(D_1) \subseteq B \setminus \acute{B}$ and $f(D_2) \subseteq \acute{B}$, but B connected, therefore there must exist at least two points $u_1 \in D_1$ and $u_2 \in D_2$ such that $f(u_1) = f(u_2) = v$, thus since $u_1 \in D_1$ then $v \in f(D_1) \subseteq B \setminus \acute{B}$, and since $u_2 \in D_2$ then $v \in f(D_2) \subseteq \acute{B}$. this implies that $v \in (B \setminus \acute{B}) \cap \acute{B} = \emptyset$, which is impossible. The contradiction shows the assumption that $f(C)$ is not dense in B was false. □

Note that the proposition only implies the density of the image not perfect equality with the range. Meaning the image $f(C)$ might not be exactly B, but it is spread out throughout B without significant gaps remain. In contrast, continuous functions preserve connectedness,

but the image of any block is not necessarily a block, as illustrated in the following proposition;

5.4 Proposition Let $(V(G_1), \tau_B(G_1))$ and $(V(G_2), \tau_B(G_2))$ be two block topological spaces. If the function $f : V(G_1) \longrightarrow V(G_2)$ is continuous, then for each block B in G_1 the image $f(B) \subseteq C$, for some block C in G_2.

Proof. Suppose that f is continuous function and B in G_1.

Case 1. If $f(B)$ contains at least two distinct vertices $u_1, u_2 \in V(G_2)$, but B is 2-connected, then its continuous image $f(B)$ is also connected. So $f(B)$ is contained in exactly one block. Otherwise, the vertices u_1 and u_2 have preimages contained within different blocks in $V(G_1)$, and this contradicts the continuity of f and 2-connectivity of blocks.

Case 2. If $f(B)$ is singleton, that proves the assertion   □

5.5 Remark In general, continuity alone is not enough to convert the blocks onto blocks. To obtain that, an adjacency preservation condition must be included;

5.6 Definition Let G_1, G_2 be connected graphs with block spaces $(V(G_1), \tau_B(G_1))$, $(V(G_2), \tau_B(G_2))$. A continuous function $f: V(G_1) \rightarrow V(G_2)$ is called block-preserving function if for every block B of G_1 and every pair of vertices $u, v \in B$, the images $f(u)$ and $f(v)$ are contained entirely in a common block of G_2.

5.7 Proposition The block space $\left(V(G), \tau_B(G)\right)$ induced by connected graph $G = (V, E)$ is contractible.

Proof. Let G be a connected graph with $|G| = n$. The block space induced by G is Alexandroff, so there exists specialization preorder defined by $c \preccurlyeq B$, for each cut-vertex c and for each block B containing c. this order is isomorphic to the block-cut tree of G, which can be viewed as finite partially order set and denoted by P_G. Now, using induction on $n = |V(G)|$:

If $n = 1$, the block space has only one point, it is trivially contractible. For $n = k > 1$, assume that every block space $\left(V(\dot{G}), \tau_B(\dot{G})\right)$ with $|V(\dot{G})| \leq k$ is contractible. Now, Let the block space with size $k + 1 = n$. And since from stong's lemma: if a is a beat point of finite poset P, then the associated topological space is homotopy equivalent to the space associated with $P \setminus \{a\}$, thus it is sufficient to prove that the poset P_G of connected graph with more than one point contains a beat point. Since P_G represent as tree, therefor it has at least one leaf. Then the leaf $\mathcal{L}$ is either;

Case 1. Maximal element, then $\mathcal{L}$ is a block B and covering a unique cut-vertex c, consequently, B is a down beat point.

Case 2. Minimal element, then $\mathcal{L}$ is a cut-vertex c and covered by a unique block B, consequently, c is an up beat point.

Thus, the poset P_G has at least one beat point a in all cases. So by stong's lemma, the block space $\left(V(G), \tau_B(G)\right)$ is homotopy equivalent to the space $\left(V(\dot{G}), \tau_B(\dot{G})\right)$ associated by $P_G \setminus \{a\}$ with k elements which is contractible from the inductive hypothesis. This implies that the block space $\left(V(G), \tau_B(G)\right)$ is contractible.   □

Contractibility and block-preserving condition implies that;

5.8 Proposition Let G be a connected graph, and $f : V(G) \longrightarrow V(G)$ be a block-preserving function on the block space $V(G)$, then f has at least one fixed point in $V(G)$.

Proof. Let $f : V(G) \longrightarrow V(G)$ be block-preserving of the connected graph G. From proposition (5.7), the block space $V(G)$ is contractible. Thus, for every $u \in V(G)$ there is a homotopy

$$H: V(G) \times [0,1] \to V(G) \quad \text{such that} \quad H(u, 0) = u \text{ and } H(u, 1) = u_0$$

For some $u_0 \in V(G)$. Now, assume, by contradiction, that f has no fixed point and define $\widetilde{H}(u, x) = H(f(u), x)$ with $\widetilde{H}(u, 0) = H(f(u), 0) = f(u)$ at $x = 0$, and $\widetilde{H}(u, 1) = H(f(u), 1) = u_0$ at $x = 1$, therefore $\widetilde{H}$ is homotopy between f and the constant map at u_0, but the identity map is homotopic to constant map, therefore f is homotopic the identity map without the diagonal $\{(u, u) : u \in V(G)\}$, but from the contractibility, every self-map homotopic the identity map must admit a fixed point, this is a contradiction. Consequently, there exists $u \in V(G)$ such that $f(u) = u$. □

5.9 Definition Two graphs G_1 and G_2 are isomorphic and denote $G_1 \cong G_2$ if there exists a bijective function $f : V(G_1) \to V(G_2)$, such that for each two adjacent vertices u and v in G_1, then $f(u)$ and $f(v)$ are also adjacent in G_2. The function f is called isomorphism.

5.10 Theorem Let G_1 and G_2 are connected graphs, and $f : V(G_1) \to V(G_2)$ be isomorphism. Then
i. f is a block-preserving function.
ii. The block spaces $(V_1, \tau_B(G_1))$ and $(V_2, \tau_B(G_2))$ are homeomorphic.

Isomorphisms guarantee homeomorphisms, but the converse is not true in general. For (ii), consider C_n and K_n for $n > 3$, their block topologies are both indiscrete, so these topologies are homeomorphic, but the two graphs are not isomorphic.

5.11 Proposition Two block spaces are homeomorphic if and only if the block-cut trees of their corresponding graphs are isometric. In particular, the corresponding blocks have the same number of vertices.
Proof. Let G_1 and G_2 be connected graphs, and $\psi : V(G_1) \to V(G_2)$ be a homeomorphism between the block spaces induced by them. Since the only singleton open sets in block space is cut-vertices, and homeomorphism preserves open sets, thus ψ sends cut-vertices bijectively onto cut-vertices. Moreover, a pure vertex $u \in V(G_1)$ contained in the minimal open set B_u, under ψ, the image $\psi(B_u)$ is the minimal open set containing $\psi(u)$, which is a unique block in G_2 containing $\psi(u)$, so $\psi(B_u) = B_{f(u)}$. This means that, a vertex v belongs to block B in G_1 if and only if $\psi(v)$ belongs to $\psi(B)$ in G_2. Thus, ψ is bijective correspondence from blocks of G_1 to those of G_2 and also it preserves the incidence. This implies existence of an isomorphism between block-cut trees of G_1 and G_2. Moreover, the number of cut-vertices incident to a block B equal that incident to $\psi(B)$, and the pure vertices contained in B correspond those in $\psi(B)$, hence the total number of vertices of the corresponding blocks are the same.
For the converse, assume that ϕ is an isomorphism between the block-cut trees of G_1 and G_2, and for each block B, let $|B| = |\phi(B)|$. Define a function $\psi : V(G_1) \to V(G_2)$ such that for each vertex $v \in V(G_1)$
1. if v is cut-vertex, set $\psi(v) = \phi(v)$, which is cut-vertex in G_2 because ϕ is bijection.
2. if v is pure vertex, it belongs to the unique block B_v, from the hypotheses, B_v and its image have the same number of vertices, and same number of incident cut-vertices to them, so same number of pure vertices. Thus, set $\psi(v)$ is the chosen image in $\phi(B_v)$, therefore, ψ is bijection sends cut-vertices onto cut-vertices and each block B onto the block $\phi(B)$ as sets, this means that, ψ is continuous and has continuous inverse. Thus ψ is a homeomorphism. □

5.12 Corollary block spaces induced by isometric graphs are symmetric [4].
Proof. Suppose that $G_1 \cong G_2$, so there is an homeomorphism $\phi : V(G_1) \to V(G_2)$ between $(V(G_1), \tau_B(G_1))$ and $(V(G_2), \tau_B(G_2))$, since ϕ is a bijective function, also it is continuous and has continuous inverse, then image each open set in $\tau_B(G_1)$ is open in $\tau_B(G_2)$, and vice versa, so the number of open sets must be the same, therefore $|\tau_B(G_1)| = |\tau_B(G_2)|$. Moreover, for

all $A \in \tau_B(G_1)$, the set $B = \phi(A) \in \tau_B(G_2)$ and have same count of elements by bijection, thus $|A| = |B|$. the converse is obvious. $\qquad\square$

Conclusion:

This paper presents a systematic study of the fundamental properties of block spaces of undirected simple graphs. Our main results include: characterizing closure points and dense sets, along with some of their special properties; providing a precise criterion for disconnectedness when bridge edges are present; demonstrating that block spaces are contractile with the possibility of removing removable points; verifying the fixed-point theorem for block-preserving functions; and finally, arriving at a complete classification of topological homeomorphism: two block spaces are topologically homeomorphic if and only if their block trees are isomorphic and the corresponding blocks are equal in size. This demonstrates that block topology preserves the connectivity skeleton of the graph without focusing on the details of its internal edges.

Compliance with ethical standards

Disclosure of conflict of interest

The authors declare that they have no conflict of interest.

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