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A Study on ƅ$^{*}_{i,j}$I-Open Set within Ideal Biopological Spaces

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دراسة حول المجموعات ƅ$^{*}_{i,j}$I المفتوحة داخل الفضاءات ثنائية التبولوجيا المثالية

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Abstract:
This paper introduces the concepts of ƅ$^{*}_{i,j}$I-open sets and ƅ$^{*}_{i,j}$I-closed sets in the framework of ideal ditopological spaces, presenting them as a new form of generalized open and closed sets. ƅ$^{*}_{i,j}$I-open sets are defined as subsets satisfying specific generalized openness conditions with respect to two topologies and an ideal, while ƅ$^{*}_{i,j}$I -closed sets are their complements. The study begins with a precise definition of these sets and explains how they extend previously developed ideas in generalized and bitopological structures. It examines their relationships with other known types of open and closed sets and highlights essential properties that characterize their behavior. Particular attention is given to how the interaction of the two topologies and the ideal shapes their structure and flexibility. The paper also introduces the related operators, ƅ$^{*}_{i,j}$I -interior and ƅ$^{*}_{i,j}$I -closure, and discusses their key properties. Overall, the study contributes to understanding openness and closedness in ideal bitopological spaces and provides a foundation for further research.

Keywords: Ideal ditopological spaces, ƅI-open sets, Interior operators, Closure operators, Generalized open sets.

الملخص
تقدم هذه الورقة مفاهيم المجموعات ƅ$^{*}_{i,j}$I - المفتوحة وƅ$^{*}_{i,j}$I - المغلقة في إطار الفضاءات ثنائية الطوبولوجيا مع وجود مثالي، باعتبارها شكلا جديدا من المجموعات المفتوحة والمغلقة في الطوبولوجيا المعممة. تعرف المجموعات المفتوحة ƅ$^{*}_{i,j}$I على انها المجموعات التي تحقق شروط الانفتاح المعمم بالنسبة لطوبولوجيين اثنين والمثالي، بينما تعرف المجموعات المغلقة على انها المكملات الخاصة بها. تبدأ الدراسة بتعريف دقيق لهذه المجموعات وشرح كيفية توسعها للأفكار المطورة سابقا في البنى الطوبولوجية الثنائية والمعممة. كما تبحث الدراسة علاقاتها بالمجموعات المفتوحة والمغلقة المعروفة وتبرز الخصائص الاساسية التي تحدد سلوكها. ويولى اهتمام خاص لتأثير تفاعل الطوبولوجيين والمثالي على بنية هذه المجموعات ومرونتها ضمن الفضاء.

وبعد ذلك تقدم الدراسة أيضا العاملين المرتبطين بهذه المجموعات، وهما $\mathfrak{b}^*_{i,j}I-\text{interior}$ و $\mathfrak{b}^*_{i,j}I-$ closure، وتناقش الخصائص الاساسية المرتبطة بهما. تساهم الدراسة في تعزيز الفهم العميق للانفتاح والانغلاق في الفضاءات ثنائية الطوبولوجيا المثالية وتوفر قاعدة للبحوث المستقبلية.

الكلمات المفتاحية: الفضاءات ثنائية التبولوجيا المثالية، المجموعات- $\mathfrak{b}I$ المفتوحة، معامل الداخل، معامل الانغلاق، المجموعات المفتوحة المعممة.

1-Introduction

Kelly [13], introduced the concept of a bitopological space $(X, \mathcal{T}_1, \mathcal{T}_2)$, where a set X is equipped with two topologies $\mathcal{T}_1$ and $\mathcal{T}_2$. In 1966, Kuratowski[14] studied the notion of ideals on a topological space $(X,\mathcal{T})$, defined as collections of subsets of X with the hereditary (i) if $\mathcal{A} \in \mathrm{I}$ and $\mathcal{B} \subset \mathcal{A}$, then $\mathcal{B} \in \mathrm{I}$ and (ii) if $\mathcal{A} \in \mathrm{I}$ and $\mathcal{B} \in \mathrm{I}$ then $\mathcal{A} \cup \mathcal{B} \in \mathfrak{I}$. Let $(X, \mathcal{T}_1, \mathcal{T}_2, \mathrm{I})$ an ideal bitopological space consider the power set $\mathbb{P}(X)$ which consists of all subsets of X.
Janković[12], define a set operator $(.)^*_{\mathrm{i}}: \mathcal{P}(X) \to \mathcal{P}(X)$ called the local function of $\mathcal{A}$ with respect to $\mathcal{T}_{\mathrm{i}}$ and I, as for $\mathcal{A} \subset X$, $\mathcal{A}^*_{\mathrm{i}}(\mathcal{T}_{\mathrm{i}}, \mathrm{I})=\{x\in X| \mathcal{B} \cap \mathcal{A} \notin$?, for every $\mathcal{B} \in \mathcal{T}_{\mathrm{i}}(x)\}$ as well as $\mathcal{T}_{\mathrm{i}}(x) = \{\mathcal{B} \in \mathcal{T}_{\mathrm{i}} | x \in \mathcal{B}\}$.For each ideal topological space $(X, \mathcal{T}_1, \mathcal{T}_2, \mathrm{I})$, there exists a topology $\mathcal{T}^*_{\mathrm{i}}$ that is finer than$\mathcal{T}_{\mathrm{i}}$, generated by the base by the base $\mathbb{B}(\mathrm{I}, \mathcal{T}) = \{\mathcal{A} \setminus \mathrm{I} | \mathcal{A} \in \mathcal{T}$ and $\mathrm{I} \in \mathrm{I}\}$, However, in general, $\mathbb{B}(\mathrm{I}, \mathcal{T})$does not always form a topology.
Furthermore, observe that the closure operator corresponding to the topology $\mathcal{T}^*_{\mathrm{i}}(\mathrm{I})$,which is finer than$\mathcal{T}_{\mathrm{i}}$, is defined by $\mathrm{Cl}^*_{\mathrm{i}}(\mathcal{A}) = \mathcal{A} \cup \mathcal{A}^*_{\mathrm{i}}$.
Accordingly, $\mathrm{Int}^*_{\mathrm{i}}(\mathcal{A})$ denotes the interior of $\mathcal{A}$ in the topology $\mathcal{T}^*_{\mathrm{i}}(\mathrm{I})$ while, $\mathrm{Int}^*_{\mathrm{i}}(\mathcal{A}^*_{\mathrm{i}})$ represents the interior of $\mathcal{A}^*_{\mathrm{i}}$with respect to the topology $\mathcal{T}^*_{\mathrm{i}}$.

The study of open sets in ideal and bitopological spaces has evolved through several key contributions. In 2007, Akdag[2] introduced the concept of $\mathfrak{b}$-I-open sets in ideal topological spaces, while in the same year, Al-Hawary and Al-Omari[3] proposed the notion of $\mathfrak{b}_{i,j}$-open sets in bitopological spaces. Building on these ideas, Sarma [15] expanded this framework by defining $\mathfrak{b}_{i,j}$I-open sets in ideal bitopological spaces. Several researchers have also discussed the concept of open sets, as presented in the studies listed in references [1], [4], [6], and [16]. These works have contributed significantly to the development of the fundamental concepts and the expansion of the theoretical framework within bitopological spaces.

2-PRELIMINARIES

In this paper, $(X, \mathcal{T}_1, \mathcal{T}_2, \mathrm{I})$ constantly mean bitopological space with no separation axioms are assumed in this space and IB-S be the abbreviation for ideal bitopological space. And assume$\mathcal{A}$ is a subset of X, then we denote respectively the interior and the closure of $\mathcal{A}$ by $\mathrm{int_i}(\mathcal{A})$ and $\mathrm{Cl_j}(\mathcal{A})$ with reference to $\mathbb{V}_i$ for $i = 1,2$.

Definition 2.1. [3] Assume $\mathcal{A}$ is a subset of the bitoplogcal space $(X, \mathcal{T}_1, \mathcal{T}_2)$ such that $\mathrm{i, j} = 1,2,$ and $\mathrm{i} \neq \mathrm{j}$. Then $\mathcal{A}$ is called:

$\mathfrak{b}_{i,j}$ -OS if $\mathcal{A} \subseteq \mathrm{Int_i}\left(\mathrm{Cl_j}(\mathcal{A})\right) \cup \mathrm{Cl_j}\left(\mathrm{Int_i}(\mathcal{A})\right)$.[1]

Definition 2.2. Assume $\mathcal{A}$ is a subset of the IB-S such that $\mathrm{i, j} = 1,2,$ and $\mathrm{i} \neq$ j.Then $\mathcal{A}$ Then $\mathcal{A}$ is called:

1- $\mathrm{I}_{i,j}$-OS if $\mathcal{A} \subset \mathrm{Int_i}\left(\mathcal{A}_\mathrm{j}^*\right)$. [8]

2- $semi_{i,j}$I-OS if $\mathrm{A} \subset \mathrm{Cl_j^*}\left(\mathrm{Int_i}(\mathcal{A})\right)$.[9]

4- $semi_{\mathrm{i,j}}^*$I-OS if $\mathcal{A} \subset \mathrm{Cl}_j\left(\mathrm{Int}_i^*(\mathcal{A})\right)$.[5]

5- $\mathrm{pre}_{i,j}$I-OS if $\mathcal{A} \subset \mathrm{Int_i}(\mathrm{Cl_j^*}(\mathcal{A}))$. [7]

7- $\mathfrak{b}_{i,j}$I -OS if $\mathcal{A} \subseteq \mathrm{Int_i}(\mathrm{Cl_j^*}(\mathcal{A})) \cup \mathrm{Cl_j^*}(\mathrm{Int_i}(\mathcal{A}))$. [15]

8- $\alpha_{i,j}$I-OS if $\mathrm{A} \subset \mathrm{Int_i}(\mathrm{Cl_j^*}\left(\mathrm{Int_i}(\mathcal{A})\right))$. [11]

Lemma2.1.[8] Let $(X, \mathcal{T}_1, \mathcal{T}_2, \mathrm{I})$ be a IB-S, and $\mathcal{A} \subset X$, then for any $(\mathrm{i, j})$I-open set we get

1- $\mathrm{Cl_j^*}(\mathcal{A}) \subset \mathrm{Cl_j}(\mathcal{A})$.

2- $\mathrm{Int_i}(\mathcal{A})) \subset \mathrm{Int_i^*}(\mathcal{A}))$.

Lemma2.2.[8] If $(X, \mathcal{T}_1, \mathcal{T}_2, \mathrm{I})$ be a IB-S, and $\mathcal{A} \subset X$, with reference to any (i,j)- I-OS then we get $\mathcal{A}_\mathrm{j}^* = (\mathrm{int_i}\left(\mathcal{A}_\mathrm{j}^*\right))_\mathrm{j}^*$.

Lemma 2.3.[10] Assume $\mathcal{A}$ is a subset of the IB-S $(X, \mathcal{T}_1, \mathcal{T}_2, \mathrm{I})$, and $\mathcal{B} \in \mathcal{T}_\mathrm{j}$. Then, $\mathrm{Cl_j}(\mathcal{A}) \cap \mathcal{B} \subseteq \mathrm{Cl_j}(\mathcal{A} \cap \mathcal{B})$.

3- ON $\mathfrak{b}_{\mathrm{i,j}}^*$I-OPEN SET

Definition 3.1. If $(X, \mathcal{T}_1, \mathcal{T}_2, \mathrm{I})$ is an ideal bitopological spaces then a subset $\mathcal{A}$ of X is called:

$\mathfrak{b}_{\mathrm{i,j}}^*$I-OS if $\mathcal{A} \subseteq \mathrm{Int_i^*}(\mathrm{Cl_j}(\mathcal{A})) \cup \mathrm{Cl_j}(\mathrm{Int_i^*}(\mathcal{A}))$,

in which $\mathrm{i, j} \in \{1,2\}, i \neq \mathrm{j}$. The collection comprised of all $\mathfrak{b}_{\mathrm{i,j}}^*$I-OS in X will be denoted by $\mathfrak{b}_{\mathrm{i,j}}^*$I-O($X$),and ꞮB-S for ideal bitopological spaces.

 |

Example 3.1. Consider IB-S such that ; X= $\{\upsilon,\mu,\omega,\tau\}$ and $Ʈ_1$= $\{\phi$,$\{\mu\}$,$\{\upsilon,\omega,\tau\}$,$X\}$, $Ʈ_2$= $\{\phi$,$\{\upsilon,\mu\}$,$X\}$.And if I = $\{\phi$,$\{\omega\}\}$. Then $\{\upsilon,\omega\}$and $\{\upsilon,\mu,\tau\}$ are $ƀ^*_{1,2}$⍰-OS, but $\{\omega,\tau\}$ is not $ƀ^*_{1,2}$I-OS.

Example3.2. Let IB-S such that ; X= $\{\upsilon,\mu,\omega\}$ and $Ʈ_1$=$\{\phi$,$\{\upsilon\}$,$\{\mu\}$,$\{\upsilon,\mu\}$,$X\}$, $Ʈ_2$= $\{\phi$,$\{\upsilon,\omega\}$,$X\}$.And if I = $\{\phi$, $\{\mu\}$,$\{\omega\}$,$\{\mu,\omega\}\}$. Then $\{\mu,\omega\}$ is $ƀ^*_{1,2}$I-OS, but $\{\mu,\omega\}$ is not $ƀ^*_{2,1}$I-OS.

Example 3.3. Let IB-S such that ; X= $\{\upsilon,\mu,\omega,\tau\}$ and $Ʈ_1$= $\{\phi$,$\{\mu\}$,$\{\upsilon,\omega\}$,$\{\upsilon,\mu,\omega\}$,$X\}$, $Ʈ_2$= $\{\phi$,$\{\omega\}$,$X\}$.And if I = $\{\phi$, $\{\upsilon\}$,$\{\mu\}$,$\{\upsilon,\mu\}\}$. Then $\{\upsilon$,$\mu,\omega\}$and $\{\upsilon,\tau\}$ are a $ƀ^*_{1,2}$I-OS, but $\{\tau\}$ is not $ƀ^*_{1,2}$I-OS.

Example 3.4. Let IB-S such that ; X= $\{\upsilon,\mu,\omega\}$ and $Ʈ_1$= $\{\phi$,$\{\omega\}$,$X\}$, $Ʈ_2$= $\{\phi$,$\{\upsilon\}$,$\{\upsilon,\omega\}$,$X\}$.And if I = $\{\phi$,$\{\mu\}\{\omega\}$,$\{\mu,\omega\}\}$.Then all subset of X are $ƀ^*_{2,1}$I-OS.

Proposition3.1. If $(X,Ʈ_1,Ʈ_2,\mathrm{I})$ is a IB-S, and $\mathcal{A}$ a subset of X. Then the following properties are observed:

1- Every $\mathrm{I}_{i,j}$-OS is $ƀ^*_{i,j}$I-OS.

2- Every $ƀ_{\mathrm{i,j}}$I-OS is $ƀ^*_{\mathrm{i,j}}$I-OS.

3- Every $\mathrm{semi}_{\mathrm{i,j}}$I-OS is $ƀ^*_{\mathrm{i,j}}$I-OS.

4- Every $\mathrm{pre}_{\mathrm{i,j}}$I-OS is $ƀ^*_{\mathrm{i,j}}$I-OS.

5- Every $\alpha_{\mathrm{i,j}}$I-OS is $ƀ^*_{\mathrm{i,j}}$I-OS.

6- Every $\mathrm{semi}^*_{\mathrm{i,j}}$I-OS is $ƀ^*_{\mathrm{i,j}}$I-OS.

Proof. (1) Assume $\mathcal{A}$ is a $\mathrm{I}_{\mathrm{i,j}}$-OS, so $\mathcal{A}\subset \mathrm{Int_i}\left(\mathcal{A}^*_\mathrm{j}\right)\subset \mathrm{Int}_i\left(\mathcal{A}^*_\mathrm{j}\cup\mathcal{A}\right)\subset$ $\mathrm{Int}_i(\mathrm{Cl}^*_\mathrm{j}(\mathcal{A}))\subset \mathrm{Int}^*_\mathrm{i}(\mathrm{Cl}^*_\mathrm{j}(\mathcal{A}))\subset \mathrm{Int}^*_\mathrm{i}(\mathrm{Cl_j}(\mathcal{A}))$, so
$\mathcal{A}\subseteq \mathrm{Int}^*_\mathrm{i}(\mathrm{Cl_j}(\mathcal{A}))\cup \mathrm{Cl_j}(\mathrm{Int}^*_\mathrm{i}(\mathcal{A}))$.
The remaining proofs are follow from the first statement.

Let $(X,Ʈ_1,Ʈ_2,\mathrm{I})$ be a IB-S. Assume $\mathcal{A}$ is a subset of X then we get this diagram:

$\mathrm{semi}^*_{\mathrm{i,j}}$I-OS
↗ ↓ ↘
I-OS → $\alpha_{\mathrm{i,j}}$I-OS → $\mathrm{semi}_{\mathrm{i,j}}$-OS → $ƀ_{\mathrm{i,j}}$I-OS → $ƀ^*_{\mathrm{i,j}}$I-OS
↘ ↑ ↗
$\mathrm{pre}_{\mathrm{i,j}}$I-OS

Figure 1: The figure illustrates the relationships between the open sets in the ideal bitopological spaces.

Generally, the opposites of proposition 3.1 is not accurate. as demonstrated by the next example.

Example 3.5. From example 3.1 we have $\{\upsilon, \omega\}$is $\mathfrak{b}^*_{1,2}$I-OS, but it is not semi$_{1,2}$I-OS.

Example 3.6. From example 3.2 we have $\{\mu, \omega\}$is $\mathfrak{b}^*_{1,2}$I-OS, but it is not $\mathfrak{b}_{i,j}$I-OS.

Theorem 3.1. If $(X, \Tau_1, \Tau_2, \mathrm{I})$ is a IB-S, let $\mathcal{A}$ be a subset of X. If $\mathcal{A}$ is $\mathfrak{b}^*_{\mathrm{i,j}}$I-OS with $\mathrm{Int}^*_{\mathrm{i}}\left(\mathrm{Cl}_{\mathrm{j}}(\mathcal{A})\right) = \emptyset$, then $\mathcal{A}$ is semi $^*_{\mathrm{i,j}}$I-OS.

Proof. Let $\mathcal{A}$ be $\mathfrak{b}^*_{\mathrm{I,j}}$I-OS then $\mathcal{A} \subseteq \mathrm{Int}^*_{\mathrm{i}}\left(\mathrm{Cl}_{\mathrm{j}}(\mathcal{A})\right) \cup \mathrm{Cl}_{\mathrm{j}}\left(\mathrm{Int}^*_{\mathrm{i}}(\mathcal{A})\right)$

$$\subseteq \mathrm{Cl}_{\mathrm{j}}\left(\mathrm{Int}^*_{\mathrm{i}}(\mathcal{A})\right).$$

Theorem 3.2. If $(X, \Tau_1, \Tau_2, \mathrm{I})$ is a IB-S, let $\mathcal{A}$, and $\mathcal{B}$ be a subsets of X. If $\mathcal{A}$ is $\mathfrak{b}^*_{i,j}$I-OS and is a pre$_{i,j}$I-OS, then $\mathcal{A} \cup \mathcal{B}$ is $\mathfrak{b}^*_{i,j}$I -OS.

Proof.

$$\begin{aligned}\mathcal{A} \cup \mathcal{B} &\subseteq \mathrm{Int}^*_{\mathrm{i}}\left(\mathrm{Cl}_{\mathrm{j}}(\mathcal{A})\right) \cup \mathrm{Cl}_{\mathrm{j}}\left(\mathrm{Int}^*_{\mathrm{i}}(\mathcal{A})\right)) \cup \mathrm{Int}_{i}\left(\mathrm{Cl}^*_{i}(\mathcal{B})\right) \\ &\subseteq \mathrm{Int}^*_{\mathrm{i}}\left(\mathrm{Cl}_{\mathrm{j}}(\mathcal{A})\right) \cup \mathrm{Cl}_{\mathrm{j}}\left(\mathrm{Int}^*_{\mathrm{i}}(\mathcal{A})\right)) \cup \mathrm{Int}^*_{\mathrm{i}}\left(\mathrm{Cl}_{\mathrm{j}}(\mathcal{B})\right) \\ &\subseteq \mathrm{Int}^*_{\mathrm{i}}\left(\mathrm{Cl}_{\mathrm{j}}(\mathcal{A}) \cup \mathrm{Cl}_{\mathrm{j}}(\mathcal{B})\right) \cup \mathrm{Cl}_{\mathrm{j}}(\mathrm{Int}^*_{\mathrm{i}}(\mathcal{A}) \cup \mathrm{Int}^*_{\mathrm{i}}(\mathcal{B})) \\ &\subseteq \mathrm{Int}^*_{\mathrm{i}}\left(\mathrm{Cl}_{\mathrm{j}}(\mathcal{A} \cup \mathcal{B})\right) \cup \mathrm{Cl}_{\mathrm{j}}(\mathrm{Int}^*_{\mathrm{i}}(\mathcal{A} \cup \mathcal{B})).\end{aligned}$$

So. $\mathcal{A} \cup \mathcal{B}$ is $\mathfrak{b}^*_{i,j}$I-OS.

Theorem 3.3. If $(X, \Tau_1, \Tau_2, \mathrm{I})$ is a IB-S, let $\mathcal{A}$, and $\mathcal{B}$ be a subsets of X. If $\mathcal{A}$ is pre$_{i,j}$I-OS and $\mathcal{B}$ is semi$^*_{i,j}$I-OS, then $\mathcal{A} \cup \mathcal{B}$ is $\mathfrak{b}^*_{i,j}$I -OS.

Proof.

$$\begin{aligned}\mathcal{A} \cup \mathcal{B} &\subseteq \mathrm{Int}_{\mathrm{i}}(\mathrm{Cl}^*_{\mathrm{j}}(\mathcal{A})) \cup \mathrm{Cl}_{\mathrm{j}}(\mathrm{Int}^*_{\mathrm{i}}(\mathcal{B})) \\ &\subseteq \mathrm{Int}_{\mathrm{i}}(\mathrm{Cl}^*_{\mathrm{j}}(\mathcal{A} \cup \mathcal{B}) \cup \mathrm{Cl}_{\mathrm{j}}(\mathrm{Int}^*_{\mathrm{i}}(\mathcal{A}) \cup \mathrm{Int}^*_{\mathrm{i}}(\mathcal{B})) \\ &\subseteq \mathrm{Int}^*_{\mathrm{i}}\left(\mathrm{Cl}_{\mathrm{j}}(\mathcal{A} \cup \mathcal{B})\right) \cup \mathrm{Cl}_{\mathrm{j}}(\mathrm{Int}^*_{\mathrm{i}}(\mathcal{A} \cup \mathcal{B})).\end{aligned}$$

So. $\mathcal{A} \cup \mathcal{B}$ is $\mathfrak{b}^*_{i,j}$I-OS.

Theorem 3.4. If $(X, \Tau_1, \Tau_2, \mathrm{I})$ is a IB-S, let $\mathcal{A}$, and $\mathcal{B}$ be a subsets of X. If $\mathcal{A}$ is $\mathfrak{b}^*_{\mathrm{i,j}}$I-OS and $\mathcal{B} \in \Tau_{\mathrm{i}}$, then $\mathcal{A} \cap \mathcal{B}$ is $\mathfrak{b}^*_{\mathrm{i,j}}$I-OS.

Proof. Let $\mathcal{A}$ be $\mathfrak{b}^*_{\mathrm{i,j}}$I-OS, so $\mathcal{A} \subseteq \mathrm{Int}^*_{\mathrm{i}}\left(\mathrm{Cl}_{\mathrm{j}}(\mathcal{A})\right) \cup \mathrm{Cl}_{\mathrm{j}}\left(\mathrm{Int}^*_{\mathrm{i}}(\mathcal{A})\right)$

Then $$\mathcal{A} \cap \mathcal{B} \subseteq \mathrm{Int}^*_{\mathrm{i}}\left(\mathrm{Cl}_{\mathrm{j}}(\mathcal{A})\right) \cup \mathrm{Cl}_{\mathrm{j}}\left(\mathrm{Int}^*_{\mathrm{i}}(\mathcal{A})\right) \cap \mathcal{B}$$

$$\subseteq \mathrm{Int}_i^*\big(\mathrm{Cl}_j(\mathcal{A}) \cap \mathcal{B}\big) \cup \mathrm{Cl}_j(\mathrm{Int}_i^*(\mathcal{A}) \cap \mathcal{B})$$

$$\subseteq \mathrm{Int}_i^*\Big(\mathrm{Cl}_j(\mathcal{A} \cap \mathcal{B})\Big) \cup \mathrm{Cl}_j\big(\mathrm{Int}_i^*(\mathcal{A} \cap \mathcal{B})\big).$$

Then. $\mathcal{A} \cap \mathcal{B}$ is $ƀ_{i,j}^*$I-OS

Theorem 3.5. If $(X, \mathrm{T}_1, \mathrm{T}_2, \mathrm{I})$ is a IB-S.Then the union of any $ƀ_{i,j}^*$I-OS is a $ƀ_{i,j}^*$I-OS.

Proof. Let $\mathcal{A}_\alpha \in ƀ_{i,j}^*$I-OS($X$) for each $\alpha \in \Lambda$,where Λ is an index set

$$\mathcal{A}_\alpha \subseteq \mathrm{Int}_i^*\Big(\mathrm{Cl}_j(\mathcal{A}_\alpha)\Big) \cup \mathrm{Cl}_j\big(\mathrm{Int}_i^*(\mathcal{A}_\alpha)\big) \text{ , thus}$$

$$\bigcup_{\alpha\in\Lambda} \mathcal{A}_\alpha \subseteq \bigcup_{\alpha\in\Lambda} \{\mathrm{Int}_i^*\Big(\mathrm{Cl}_j(\mathcal{A}_\alpha)\Big) \cup \mathrm{Cl}_j\big(\mathrm{Int}_i^*(\mathcal{A}_\alpha)\big)\}$$

$$\subseteq \{\mathrm{Int}_i^*(\bigcup_{\alpha\in\Lambda}\mathrm{Cl}_j(\mathcal{A}_\alpha)) \cup (\mathrm{Cl}_j(\bigcup_{\alpha\in\Lambda}\mathrm{Int}_i^*(\mathcal{A}_\alpha)))\}$$

$$\subseteq \{\mathrm{Int}_i^*(\mathrm{Cl}_j(\bigcup_{\alpha\in\Lambda} \mathcal{A}_\alpha)) \cup \mathrm{Cl}_j(\mathrm{Int}_i^*(\bigcup_{\alpha\in\Lambda} \mathcal{A}_\alpha))\}.$$

Then $\bigcup_{\alpha\in\Lambda} \mathcal{A}_\alpha$ is $ƀ_{i,j}^*$I-OS.

Definition 3.2. If $(X, \mathrm{T}_1, \mathrm{T}_2, \mathrm{I})$ is a IB-S, let $\mathcal{A}$ be a subset of X $\mathcal{A}$ is called a $ƀ_{i,j}^*$I-closed set if it is complement is a $ƀ_{i,j}^*$I-OS. All $ƀ_{i,j}^*$I-CS in X will denoted by $ƀ_{i,j}^*$I-C(X).

Theorem3.6. If $(X, \mathrm{T}_1, \mathrm{T}_2, \mathrm{I})$ is a IB-S, $\mathcal{A} \subset X$, then $\mathcal{A}$ is a $ƀ_{i,j}^*$I-CS if,
$\mathrm{Cl}_i^*(\mathrm{Int}_j(\mathcal{A})) \cap \mathrm{Int}_j(\mathrm{Cl}_i^*(\mathcal{A})) \subseteq \mathcal{A}$.
Proof. Assume $\mathcal{A}$ is a $ƀ_{i,j}^*$I-CS of $(X, \mathrm{T}_1, \mathrm{T}_2, \mathrm{I})$, then ($X$ -$\mathcal{A}$) is $ƀ_{i,j}^*$I-OS and since

$$X - \mathcal{A} \subseteq \mathrm{Int}_i\Big(\mathrm{Cl}_j^*(X - \mathcal{A})\Big) \cup \mathrm{Cl}_j^*\big(\mathrm{Int}_i(X - \mathcal{A})\big)$$

$$\subseteq \mathrm{Int}_i\Big(\mathrm{Cl}_j(X - \mathcal{A})\Big) \cup \mathrm{Cl}_j\big(\mathrm{Int}_i(X - \mathcal{A})\big)$$

$$\subseteq \{X - \mathrm{Cl}_i^*(\mathrm{Int}_j(\mathcal{A}))\} \cup \{X - \Big(\mathrm{Int}_i(\mathrm{Cl}_j^*(\mathcal{A})\Big))$$

$$\subseteq X - \{\mathrm{Cl}_i^*(\mathrm{Int}_j(\mathcal{A})) \cap \mathrm{Int}_j(\mathrm{Cl}_i^*(\mathcal{A}))\}, \text{ so}$$

$$\mathrm{Cl}_i^*(\mathrm{Int}_j(\mathcal{A})) \cap \mathrm{Int}_j(\mathrm{Cl}_i^*(\mathcal{A})) \subseteq \mathcal{A}.$$

Theorem 3.7. If $(X, \mathrm{T}_1, \mathrm{T}_2, \mathrm{I})$ is a IB-S, let $\mathcal{A}$, and $\mathcal{B}$ be a subsets of X. If $\mathcal{A}$, and $\mathcal{B}$ are $ƀ_{i,j}^*$I-CS, then $\mathcal{A} \cap \mathcal{B}$ is $ƀ_{i,j}^*$I-CS.
Proof. Let $\mathcal{A}$, and $\mathcal{B}$ be $ƀ_{i,j}^*$I-CS, so

$$\{\mathrm{Cl}_i^*(\mathrm{Int}_j(\mathcal{A})) \cap \mathrm{Int}_j(\mathrm{Cl}_i^*(\mathcal{A}))\} \cap \{\mathrm{Cl}_i^*(\mathrm{Int}_j(\mathcal{B})) \cap \mathrm{Int}_j(\mathrm{Cl}_i^*(\mathcal{B}))\} \subseteq \mathcal{A} \cap \mathcal{B}$$

$$\{\mathrm{Cl}_i^*(\mathrm{Int}_j(\mathcal{A})) \cap \mathrm{Cl}_i^*(\mathrm{Int}_j(\mathcal{B}))\} \cap \{\mathrm{Int}_j(\mathrm{Cl}_i^*(\mathcal{A})) \cap \mathrm{Int}_j(\mathrm{Cl}_i^*(\mathcal{B}))\} \subseteq \mathcal{A} \cap \mathcal{B}$$

$$\mathrm{Cl}_i^*(\mathrm{Int}_j(\mathcal{A} \cap \mathcal{B})) \cap \mathrm{Int}_j(\mathrm{Cl}_i^*(\mathcal{A} \cap \mathcal{B})) \subseteq \mathcal{A} \cap \mathcal{B}$$

Definition 3.3. If $(X, \Tau_1, \Tau_2, \mathrm{I})$ is a IB-S, let $\mathcal{A}$ be a subset of X, and x be a point of X. Then,

1- x is called an $b^*_{i,j}$I-interior point of $\mathcal{A}$ if there exists $\mathcal{B} \in b^*_{i,j}$I-OS such that $x \in \mathcal{B} \subset \mathcal{A}$.

2- The set of all $b^*_{i,j}$I-interior points of $\mathcal{A}$ is called $b^*_{i,j}$I-interior of $\mathcal{A}$ and is denoted by $b^*_{i,j}$I-Int($\mathcal{A}$).

Theorem 3.8. Let $(X, \Tau_1, \Tau_2, \mathrm{I})$ be a IB-S, and let $\mathcal{A}$ and $\mathcal{B}$ be subsets of $(X, \Tau_1, \Tau_2, \mathrm{I})$. The following properties are observed.

1- $b^*_{i,j}$I-Int($\mathcal{A}$) $= \cup \{\mathcal{U}: \mathcal{U} \subset \mathcal{A}$ and $\mathcal{A} \in b^*_{i,j}$I-O($X$)}.

2- $b^*_{i,j}$I- Int($\mathcal{A}$) is the largest $b^*_{i,j}$I-OS subset of X contained in $\mathcal{A}$.

3- $\mathcal{A}$ is $b^*_{i,j}$I-OS if and only if $\mathcal{A} = b^*_{i,j}$I-Int($\mathcal{A}$).

4- $b^*_{i,j}$I-Int($b^*_{i,j}$I- Int($\mathcal{A}$)) = $b^*_{i,j}$I- Int($\mathcal{A}$).

5- If $\mathcal{A} \subset \mathcal{B}$, then $b^*_{i,j}$I-Int($\mathcal{A}$)$\subset$ $b^*_{i,j}$I- Int($\mathcal{B}$).

6- $b^*_{i,j}$I- Int($\mathcal{A} \cap \mathcal{B}$) $\subset b^*_{i,j}$I- Int($\mathcal{A}$) $\cap$ $b^*_{i,j}$I- Int($\mathcal{B}$).

7-$b^*_{i,j}$I- Int($\mathcal{A}$) $\cup$ $b^*_{i,j}$I- Int($\mathcal{B}$) $\supset$ $b^*_{i,j}$I- Int($\mathcal{A}$) $\cup$ $b^*_{i,j}$I- Int($\mathcal{B}$).

Proof. (1) Let x $\in \cup \{\mathcal{U}: \mathcal{U} \subset \mathcal{A}$ and $\mathcal{A} \in b^*_{i,j}$I-O($X$)},then there exist $\mathcal{U} \subset b^*_{i,j}$I-O($X$), such that x $\in \mathcal{U} \subset \mathcal{A}$. Since x $\in$ $b^*_{i,j}$I-Int($\mathcal{A}$).We get $x \in \cup \{\mathcal{U}: \mathcal{U} \subset \mathcal{A}$ *and* $\mathcal{A} \in b^*_{i,j}$I-O($X$)} $\subset b^*_{i,j}$I-Int($\mathcal{A}$). In the reverse direction, the argument proceeds as follows.

x $\in b^*_{i,j}\mathrm{I} - \mathrm{Int}(\mathcal{A})$, then there exist $\mathcal{U} \subset b^*_{i,j}$I-O($X$), such that x $\in \mathcal{U} \subset \mathcal{A}$. Then $\in \cup \{\mathcal{U}: \mathcal{U} \subset \mathcal{A}$ and $\mathcal{A} \in b^*_{i,j}$I-O($X$)}.So $b^*_{i,j}\mathrm{I} - \mathrm{Int}(\mathcal{A}) \subset \cup \{\mathcal{U}: \mathcal{U} \subset$ $\mathcal{A}$ and $\mathcal{A} \in b^*_{i,j}$I-O($X$)}}, then we get $b^*_{i,j}$I-Int($\mathcal{A}$) $= \cup \{\mathcal{U}: \mathcal{U} \subset \mathcal{A}$ and $\mathcal{A} \in b^*_{i,j}$I-O($X$)},

(6) Since $\mathcal{A} \cap \mathcal{B} \subset \mathcal{A}$ and $\mathcal{A} \cap \mathcal{B} \subset \mathcal{B}$, by (4), we have $b^*_{i,j}$I-Int($\mathcal{A} \cap \mathcal{B}$) $\subset b^*_{i,j}$I-Int($\mathcal{A}$) and $b^*_{i,j}$I-Int($\mathcal{A} \cap \mathcal{B}$) $\subset b^*_{i,j}$I-Int($\mathcal{B}$).

So $b^*_{i,j}\mathrm{I} - \mathrm{Int}(\mathcal{A} \cap \mathcal{B}) \subset b^*_{i,j}\mathrm{I} - \mathrm{Int}(\mathcal{A}) \cap b^*_{i,j}\mathrm{I} - \mathrm{Int}(\mathcal{B})$.

(7). We have $b^*_{i,j}$I-Int($\mathcal{A}$) $\subset b^*_{i,j}$I-Int($\mathcal{A} \cup \mathcal{B}$) and $b^*_{i,j}$I-Int($\mathcal{B}$) $\subset b^*_{i,j}$I-Int($\mathcal{A} \cup \mathcal{B}$), then, we have $b^*_{i,j}$IInt($\mathcal{A}$) $\cup$ $b^*_{i,j}$I-Int($\mathcal{B}$) $\subset b^*_{i,j}$I-Int($\mathcal{A} \cup \mathcal{B}$).

The remaining proofs can be established directly.

Definition 3.4. Let $(X, \mathrm{T}_1, \mathrm{T}_2, \mathrm{I})$ be a IB-S, and $\mathcal{A} \subset X$, and x be a point of X. Then,

1- x is called an $\mathrm{b}^*_{i,j}$I-cluster point of $\mathcal{A}$ if $\mathcal{A} \cap \mathcal{B} \neq \phi$ for every $\mathcal{B} \in \mathrm{b}^*_{i,j}$I- OS (X, x).

2- The set of all $\mathrm{b}^*_{i,j}$I-cluster point of $\mathcal{A}$ is called $\mathrm{b}^*_{i,j}$I-cluster of $\mathcal{A}$ and is denoted by $\mathrm{b}^*_{i,j}$I-Cl$(\mathcal{A})$.

Theorem 3.9. Let $(X, \mathrm{T}_1, \mathrm{T}_2, \mathrm{I})$ be a IB-S, and let $\mathcal{A}$ and $\mathcal{B}$ be subsets of $(X, \mathrm{T}_1, \mathrm{T}_2, \mathrm{I})$. The following properties are observed:

1- $\mathrm{b}^*_{i,j}$I-Cl$(\mathcal{A}) = \bigcap \{\mathcal{U}: \mathcal{A} \subset \mathcal{U}$ and $\mathcal{U} \in \mathrm{b}^*_{i,j}$I-C$(\mathbb{X})\}$.

2- $\mathrm{b}^*_{i,j}$I-Cl$(\mathcal{A})$ is the smallest $\mathrm{b}^*_{i,j}$I- closed subset of X containing $\mathcal{A}$.

3- $\mathcal{A}$ is $\mathrm{b}^*_{i,j}$I-closed if and only if $\mathcal{A} = \mathrm{b}^*_{i,j}$I-Cl$(\mathcal{A})$.

4- $\mathrm{b}^*_{i,j}$I-Cl$(\mathrm{b}^*_{i,j}$I-Cl$(\mathcal{A})) = \mathrm{b}^*_{i,j}$I-Cl$(\mathcal{A})$.

5- If $\mathcal{A} \subset \mathcal{B}$ then $\mathrm{b}^*_{i,j}$I-Cl$(\mathcal{A}) \subset \mathrm{b}^*_{i,j}$I-Cl$(\mathcal{B})$.

6- $\mathrm{b}^*_{i,j}$I-Cl$(\mathcal{A} \cap \mathcal{B}) \subset \mathrm{b}^*_{i,j}$I-Cl$(\mathcal{A}) \cap \mathrm{b}^*_{i,j}$I-Cl$(\mathcal{B})$.

7-$\mathrm{b}^*_{i,j}$I- Cl$(\mathcal{A} \cup \mathcal{B}) \supset \mathrm{b}^*_{i,j}$I-Cl$(\mathcal{A}) \cup \mathrm{b}^*_{i,j}$I-Cl$(\mathcal{B})$.

Proof. The proofs can be established directly.

Theorem 3.10. Let $(X, \mathrm{T}_1, \mathrm{T}_2, \mathrm{I})$ be a IB-S, and let $\mathcal{A}$ be subsets of $(X, \mathrm{T}_1, \mathrm{T}_2, \mathrm{I})$, the following properties are observed:

1-$\mathrm{b}^*_{i,j}$I-Int$(X - \mathcal{A}) = X - \mathrm{b}^*_{i,j}$I-Cl$(\mathcal{A})$.

2-$\mathrm{b}^*_{i,j}$I-Cl$(X - \mathcal{A}) = X - \mathrm{b}^*_{i,j}$I-Int$(\mathcal{A})$.

Proof. (1) Let x $\in \mathrm{b}^*_{i,j}$I-Cl$(\mathcal{A})$, then there exist $\mathcal{B} \subset \mathrm{b}^*_{i,j}$I- OS$(X)$, such that $\mathcal{A} \cap \mathcal{B} = \emptyset$.We get x $\in \mathrm{b}^*_{i,j}$I-Int$(X - \mathcal{A})$,so $X - \mathrm{b}^*_{i,j}$I-Cl$(\mathcal{A}) \subset \mathrm{b}^*_{i,j}$I-Int$(X - \mathcal{A})$.
Let x $\in \mathrm{b}^*_{i,j}$I-Int$(X - \mathcal{A})$,we know $\mathrm{b}^*_{i,j}$I-Int$(X - \mathcal{A}) \cap \mathcal{B} = \emptyset$, x $\notin \mathrm{b}^*_{i,j}$I-Cl$(\mathcal{A})$, then x $\in X - \mathrm{b}^*_{i,j}$I-Cl$(\mathcal{A})$. Therefore $\mathrm{b}^*_{i,j}$I-Int$(X - \mathcal{A}) = X - \mathrm{b}^*_{i,j}$I-Cl$(\mathcal{A})$.

(2). This result can be deduced from (1).

Conclusion

In conclusion, this study has introduced and explored the concepts of $\mathrm{b}^*_{i,j}$I-open and $\mathrm{b}^*_{i,j}$I- closed sets within the framework of ideal bitopological spaces. By formulating the $\mathrm{b}^*_{i,j}$I-interior and $\mathrm{b}^*_{i,j}$I-closure, it has deepened the understanding of generalized open and closed sets under dual topologies with an ideal structure. The findings extend existing results and provide a foundation for future research on continuity, compactness, and separation axioms in ideal bitopological systems.

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