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Numerical Solution of the Simple Pendulum Equation Using the Fourth-Order Runge-Kutta Method: A Comparative Study with Analytical Solutions

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الحل العددي لمعادلة البندول البسيط باستخدام طريقة "رونج-كوتا" من الرتبة الرابعة: دراسة مقارنة مع الحلول التحليلية

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Abstract:
This study investigates the numerical solution of the non-linear pendulum differential equation using the fourth-order Runge-Kutta method. While the simple pendulum is a classic problem in dynamics, it often lacks an explicit analytical solution for arbitrary amplitudes, necessitating numerical approaches. By employing the fourth-order Runge-Kutta algorithm, this research computes the angular displacement and velocity of a pendulum with a length of 4 meters under specific initial conditions. The study performs a comprehensive comparison between the numerical results and analytical solutions to evaluate the accuracy and stability of the method. The implementation was conducted using MATLAB, where the impact of various parameters, including time range and step size (h), were systematically analyzed to optimize performance and precision. Results demonstrate that smaller step sizes (e.g., h=0.001) significantly enhance numerical accuracy, although they require increased computational time. Furthermore, the study examines energy conservation, illustrating the periodic exchange between kinetic and potential energy in the system. Ultimately, this research confirms that the fourth-order Runge-Kutta method is a highly effective and robust tool for solving second-order ordinary differential equations in physical systems. The findings provide valuable insights for applications in horology, gravimetry, and inertial navigation, where precise pendulum modeling is essential.

Keywords: Runge-Kutta Method, Numerical Analysis, Simple Pendulum, Differential Equations, MATLAB, Energy Conservation.

الملخص

تستقصي هذه الدراسة الحل العددي لمعادلة البندول غير الخطية باستخدام طريقة "رونج-كوتا" من الرتبة الرابعة .على الرغم من أن البندول البسيط يعد مسألة كلاسيكية في الديناميكا، إلا أنه غالباً ما يفتقر إلى حل تحليلي صريح للسعات العشوائية، مما يستدعي استخدام الأساليب العددية .من خلال تطبيق خوارزمية "رونج-كوتا" من الرتبة الرابعة، قام هذا البحث بحساب الإزاحة الزاوية والسرعة لبندول طوله 4 أمتار تحت شروط ابتدائية محددة .أجرت الدراسة مقارنة شاملة بين النتائج العددية والحلول التحليلية لتقييم دقة واستقرار الطريقة .تم التنفيذ باستخدام برنامجMATLAB ، حيث تم تحليل تأثير متغيرات مختلفة، بما في ذلك النطاق الزمني وحجم الخطوة(h) ، بشكل منهجي لتحسين الأداء والدقة .تظهر النتائج أن أحجام الخطوات الأصغر (مثلh=0.001) تعزز الدقة العددية بشكل كبير، على الرغم من أنها تتطلب وقتاً حسابياً أطول .علاوة على ذلك، فحصت الدراسة حفظ الطاقة، موضحة التبادل الدوري بين الطاقة الحركية والطاقة الكامنة في النظام .وفي النهاية، تؤكد هذه الدراسة أن طريقة "رونج-كوتا" من الرتبة الرابعة هي أداة فعالة وقوية للغاية لحل المعادلات التفاضلية العادية من الرتبة الثانية في الأنظمة الفيزيائية .توفر النتائج رؤى قيمة للتطبيقات في قياس الزمن، وقياس الجاذبية، والملاحة بالقصور الذاتي، حيث تعد النمذجة الدقيقة للبندول أمراً ضرورياً .

الكلمات المفتاحية: طريقة رونج-كوتا، التحليل العددي، البندول البسيط، المعادلات التفاضلية، MATLAB، حفظ الطاقة.

Introduction

The pendulum has long served as a fundamental model in physics, playing a critical role in the historical development of dynamics and time-keeping technology. While its motion is often simplified through the small-angle approximation—resulting in linear harmonic oscillation—the actual behavior of a pendulum is governed by a non-linear differential equation. Accurately describing this motion requires robust mathematical tools, especially when considering systems where the amplitude is significant or energy dissipation is involved.

Numerical methods have become indispensable in solving such non-linear ordinary differential equations (ODEs), as they provide solutions where analytical methods often fall short. Among these, the fourth-order Runge-Kutta (RK4) method is widely recognized for its efficiency and precision in approximating the solutions to initial value problems. According to Burden and Faires (2015), the accuracy of these numerical algorithms is essential for minimizing errors in dynamic simulations. Similarly, Butcher (2016) highlights that RK methods based on Taylor expansions offer superior performance compared to simpler numerical approaches.

Despite the theoretical simplicity of the pendulum, investigating its energy conservation and oscillation period through numerical simulation provides profound insights into the stability of physical systems. This paper aims to solve the non-linear differential equation of the pendulum using the fourth-order Runge-Kutta method. Furthermore, it provides a comparative analysis between the generated numerical solutions and the analytical solutions under various conditions, thereby evaluating the accuracy and reliability of the RK4 algorithm in modeling pendulum dynamics.

1. The Problem Statement

The problem is to Find the numerical solution of the differential equation of the Pendulum using Runge-Kutta Method, and compare this numerical solution to the analytical solution. The equation of the pendulum is

$$\frac{d^2\theta}{dt^2}+\frac{g}{l}\sin\theta=0 \tag{1}$$

Where g is the gravitational acceleration (9.8 m/s^2) and l is, 4 m, the length of the pendulum. The initial conditions are

1) at $t=0$, $\theta=0.01$.
2) at $t=0$, $\omega=0$.

The analytical solution of the pendulum equation is

$$\theta(t)=\theta_0\cos(\sqrt{\frac{g}{l}}t) \tag{2}$$

It desired to compute the energy and plot the $\theta(t)$.

2. Theory

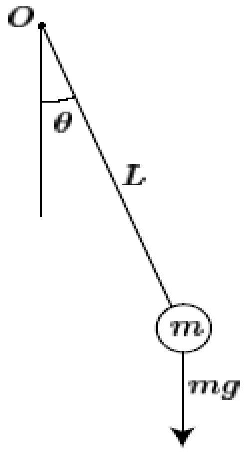
We will mention the pendulum as a physical system and its mathematical treatment. As well as the algorithm of Runge-Kutta Method.

2.1 Pendulum

A pendulum is a rigid body suspended from a fixed point (hinge) which is offset with respect to the body's center of mass. If all the mass is assumed to be concentrated at a point, we obtain the idealized simple pendulum. Pendulums have played an important role in the history of dynamics. Galileo identified the pendulum as the first example of synchronous motion, which led to the first successful clock developed by Huygens. This clock incorporated a feedback mechanism that injected energy into the oscillations (the escapement, a mechanism used in timepieces to control movement and to provide periodic energy impulses to a pendulum or balance) to compensate for friction loses. In addition to horology (the science of measuring time), pendulums have important applications in gravimetry (the measurement of the specific gravity) and inertial navigation.

Simple Pendulum

Consider a simple pendulum of mass m and length L. The equation of motion can be derived from the conservation of angular momentum about the hinge point, O,



$$I_O\ddot{\theta} = -mgL\sin\theta \ . \quad (3)$$

Since the moment of inertia is simply $I_O = mL^2$, we obtain the following non-linear equation of motion,

$$\ddot{\theta} + \frac{g}{L}\sin\theta = 0 \ . \quad (4)$$

Multiplying this equation by θ, we can write,

$$\frac{d}{dt}\left(\frac{1}{2}\dot{\theta}^2 - \frac{g}{L}\cos\theta\right) = 0, \quad (5)$$

which implies that $\dot{\theta}^2 - (2g/L)\cos\theta =$ constant. Setting $\theta = \theta_{max}$, when $\dot{\theta} = 0$ we have,

$$\dot{\theta} = \pm\sqrt{\frac{2g}{L}}\sqrt{(\cos\theta - \cos\theta_{max})} \ . \quad (6)$$

This equation cannot be integrated further in an explicit manner. Its solution must be expressed in terms of, so called, elliptic functions. The period of the oscillation, T , is obtained by multiplying by four the time it takes for the pendulum to go from $\theta = 0$ to $\theta = \theta_{max}$. Thus,

$$T = \frac{4}{\sqrt{(2g/L)}}\int_0^{\theta_{max}} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_{max}}} \ . \quad (7)$$

If we assume that the amplitude of pendulum's oscillation is small, then $\sin\theta = \theta$, and the equation of motion, given by (4), becomes linear,

$$\ddot{\theta} + \frac{g}{L}\theta = 0 \ . \quad (8)$$

This expression is much simpler than equation (4), and has solutions of the form,

$$\theta = A\sin\omega_n t + B\cos\omega_n t \ ,$$

where $\omega_n = \sqrt{g/L}$ is the natural frequency of oscillation. It is clear that these

solutions are periodic, and the period is given by

$$T = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{L}{g}} \ . \quad (9)$$

Setting $\theta = \theta_{max}$ for t = 0, we obtain,

$$\theta = \theta_{max}\cos(\omega_n t) \quad (10)$$

We observe that, in the small amplitude case, the period is independent of θ_{max}. This is called synchronism and is central to time-keeping functions in clocks. This means that, provided the amplitude is small, small changes in amplitude due to friction or other disturbances have little effect on the period.
Now one find the angular velocity ,

$\omega(t) = -\theta_{max} \sin(\omega_n t)$,

and the tangent velocity is

$$v(t) = L\,\omega(t) = -L\,\theta_{max} \sin(\omega_n t). \quad (11)$$

The total energy at any moment is the sum of potential and kinetic energy,

$$E(t) = KE(t) + PE(t)$$

Where,

$$KE = \frac{1}{2} m v^2 \quad (12)$$

Substituting v from equation (11) in equation (12), we obtain

$$KE = \frac{1}{2} m L^2 \theta_{\max}^2 \sin^2(\omega_n t) \quad (13)$$

The potential energy is related to the difference in height as

$$PE = mgL(\cos(\theta) - 1) \quad (14)$$

Where θ can be obtained from equation (10).

Summing equation (13) and equation (14), we obtain

$$E(t) = \frac{1}{2} m L^2 \theta_{\max}^2 \sin^2(\omega_n t) + mgL(\cos(\theta_{\max} \cos(\omega_n t)) - 1) \quad (15)$$

2.2 Algorithm of Runge-Kutta Method

This method is a powerful procedure, with its 4th order, that can be used to solve ordinary differential equations(ODE) , of initial value problem, numerically. Runge-Kutta (RK) methods are based on Taylor expansion formulae, but yield in general better algorithms for solutions of an ODE.

Let's consider, for example, a second-order equation of the form

$$F(t, y, y', y'') = 0 \quad (15)$$

Subject to the boundary conditions

$$y(a) = y_0,\ y'(a) = y'_0$$

when we change y' to v in equation (15), one can write it again in two forms;

$$y'' = v' = g(t, y, v), \tag{16}$$

and

$$f(v) = v \tag{17}$$

if we discretise the domain t , y , and v, we will get

$$h = \Delta t,$$

$$t_i = t_0 + ih,$$

So, equations (16) and (17) will be

$$y_i'' = v_i' = g(t_i, y_i, v_i), \tag{18}$$

And

$$f(v_i) = v_i, \tag{19}$$

$$so\ v_0 = y_0$$

Now we can apply the Runge-Kutta of 4th order by using eaquation (18) to obtain v and equation (19) to find the solution of y, as

$$v_{i+1} = v_i + \frac{1}{6}(k'_{1,i} + 2k'_{2,i} + 2k'_{3,i} + k'_{4,i}) \tag{20}$$

$$y_{i+1} = y_i + \frac{1}{6}(k_{1,i} + 2k_{2,i} + 2k_{3,i} + k_{4,i}) \tag{21}$$

Where each k or k' are number that can be found as

$$k'_{1,i} = hg(t_i, y_i, v_i),$$

$$k'_{2,i} = hg(t_i + \frac{h}{2}, y_i + \frac{k'_{1,i}}{2}, v_i + \frac{k'_{1,i}}{2}),$$

$$k'_{3,i} = hg(t_i + \frac{h}{2}, y_i + \frac{k'_{2,i}}{2}, v_i + \frac{k'_{2,i}}{2}),$$
$$k'_{4,i} = hg(t_i + h, y_i + k'_{3,i}, v_i + k'_{3,i}),$$
and so on
$$k_{1,i} = hf(v_i),$$
$$k_{2,i} = hf(v_i + \frac{k_{1,i}}{2}),$$
$$k_{3,i} = hf(v_i + \frac{k_{2,i}}{2}),$$
$$k_{4,i} = hf(v_i + k_{3,i}).$$

3. Calculations

The solution equation (1) with the boundary condition by using Runge-Kutta methods require to write the following

$$\frac{d^2\theta}{dt^2} = g(\theta) = -\frac{g}{l}\sin\theta,$$
$$f(v) = f(\omega) = \omega,$$

Where g is the gravitational acceleration (9.8 m/s^2) and l is, 4 m, the length of the pendulum. The initial conditions are
- at $t=0$, $\theta_0=0.01$.
- at $t=0$, $\omega_0=0$.

$$g(\theta_i) = -\frac{9.8}{4}\sin\theta_i,$$
$$f(\omega_i) = \omega_i,$$

The runge-kutta equations will be

$$\omega_{i+1} = \omega_i + \frac{1}{6}(k'_{1,i} + 2k'_{2,i} + 2k'_{3,i} + k'_{4,i}) \quad (22)$$

$$\theta_{i+1} = \theta_i + \frac{1}{6}(k_{1,i} + 2k_{2,i} + 2k_{3,i} + k_{4,i}) \quad (23)$$

Where each k or k' are number that can be found as

$$k'_{1,i} = hg(\theta_i) = h\frac{-9.8}{4}\sin\theta_i,$$

$$k'_{2,i} = hg(\theta_i + \frac{k'_{1,i}}{2}) = h\frac{-9.8}{4}\sin(\theta_i + \frac{k'_{1,i}}{2}),$$

$$k'_{3,i} = hg(\theta_i + \frac{k'_{2,i}}{2}) = h\frac{-9.8}{4}\sin(\theta_i + \frac{k'_{2,i}}{2}),$$

$$k'_{4,i} = hg(\theta_i + k'_{3,i}) = h\frac{-9.8}{4}\sin(\theta_i + k'_{3,i}),$$

$$\textit{and so on}$$

$$k_{1,i} = hf(\omega_i) = h\omega_i,$$

$$k_{2,i} = hf(\omega_i + \frac{k_{1,i}}{2}) = h(\omega_i + \frac{k_{1,i}}{2}),$$

$$k_{3,i} = hf(\omega_i + \frac{k_{2,i}}{2}) = h(\omega_i + \frac{k_{2,i}}{2}),$$

$$k_{4,i} = hf(\omega_i + k_{3,i}) = h(\omega_i + k_{3,i}).$$

It is clear now how to calculate the equations of our problem in (22) and (23). We just need to specify the the step size *h* then we start setting i=0, 1.. and so on. We will do procedure in a program written in MATLAB.

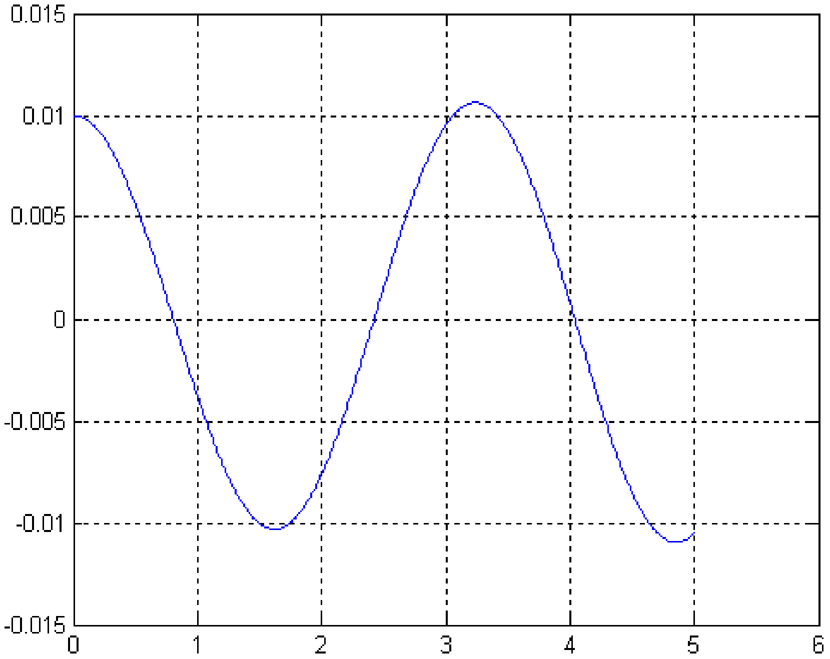
4. Results

The program for the calculations is written in MATLAB. It is made for our problem only. Here we need to investigate the results with changing many parameters in our calculation. As a numerical solution, the step size may have a clear role in the results, and this step size may relate to the range of time t when it is taken large. For a simple harmonic oscillator sinθ is approximated to θ when θ is very small.

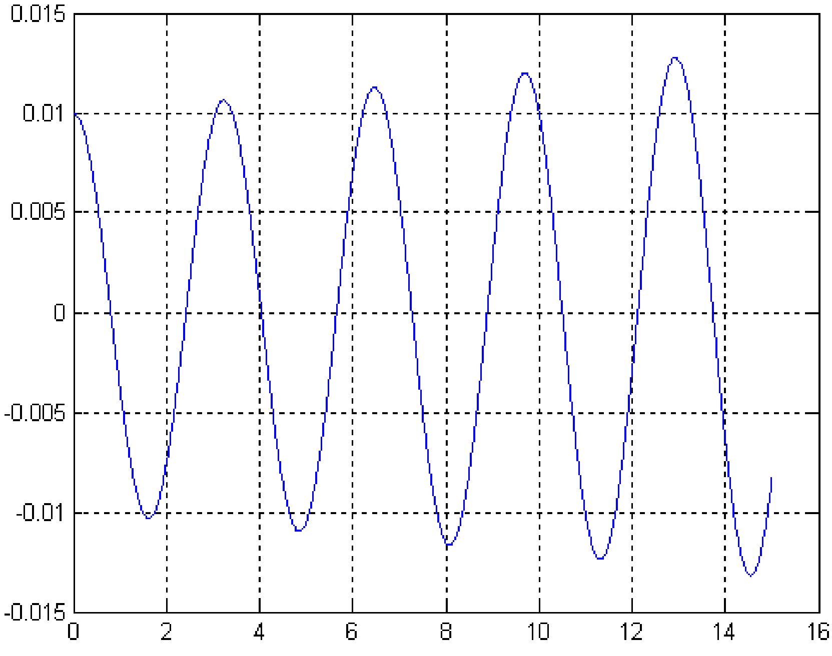
Before we start, it is important to mention that
L=4m, g=-9.8 and our equation is with sinθ.

a- investigation the time range t= 0 to 5

we need to consider the other parameters constant, for example,
θ_0=0.01
h=0.01

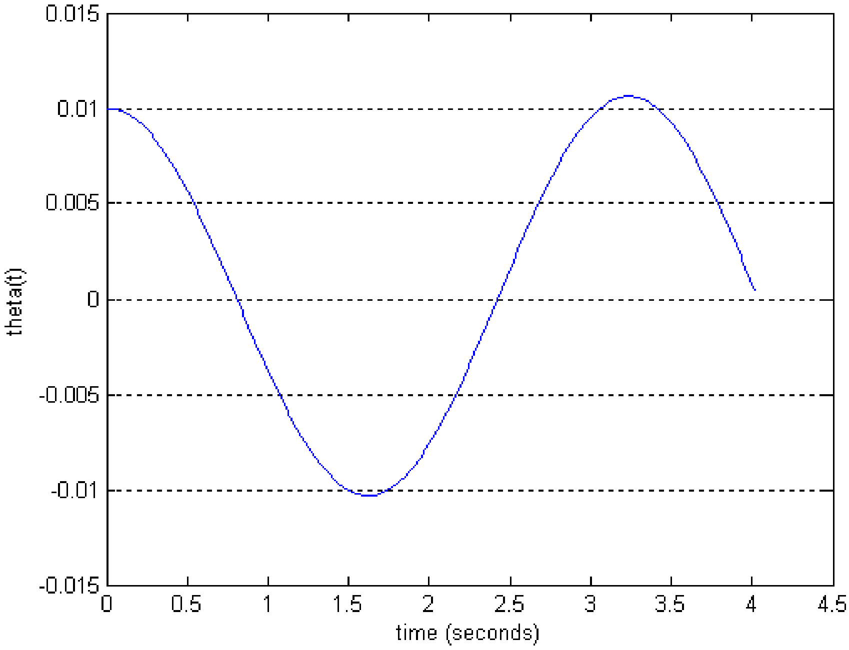


b- the time range t= 0 to 15 (h=0.01)

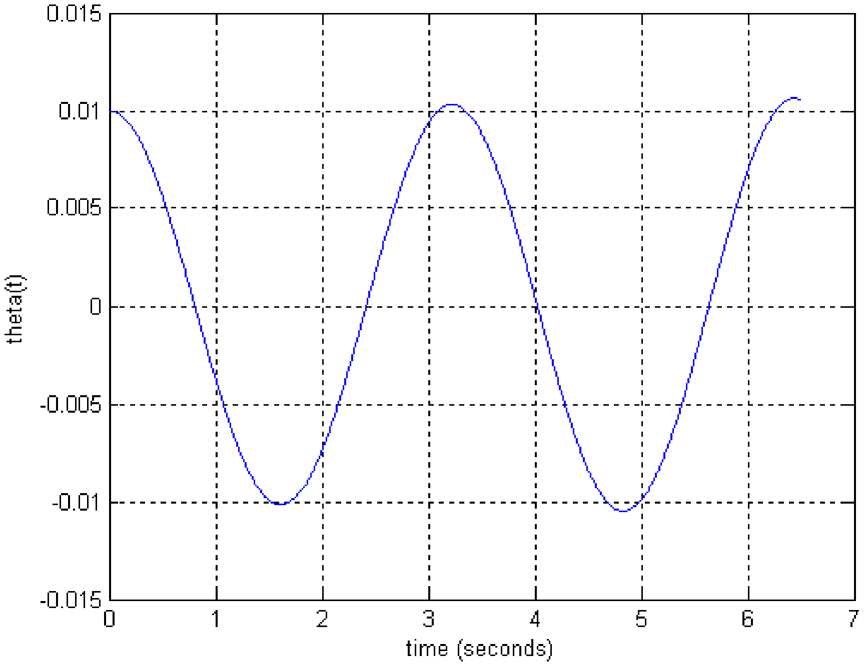


It is clear we have overestimate after time 1 sec.

c- the time range t= 0 to T (period) (h=0.01)
T=4.0142 seconds

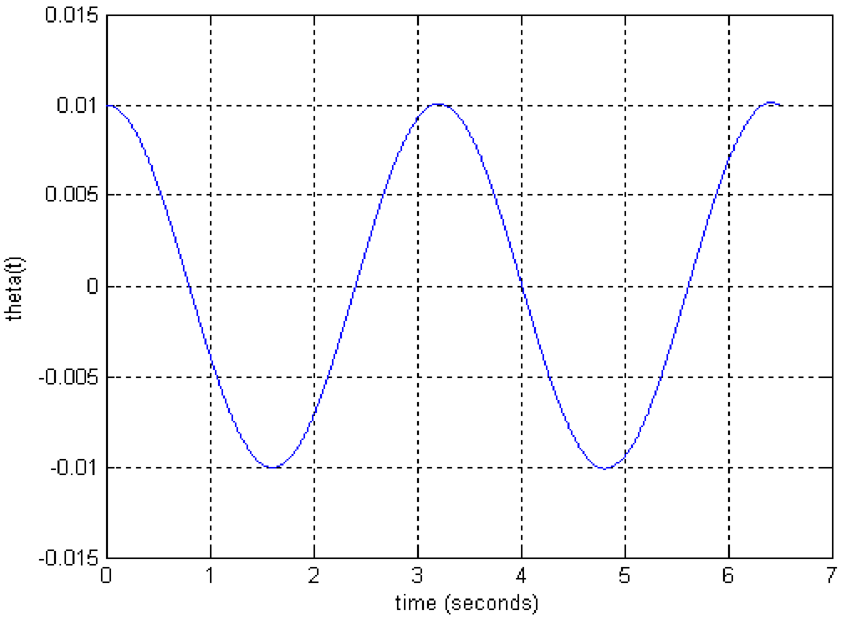


It is better to take time t=0 to 6.5sec.
d- The time range t= 0 to 6.5 and investigation of (h=0.005)



If we compare the peak at t=3 sec, the overestimate is less.

e- **The time range t= 0 to 6.5, and investigating h=0.001**

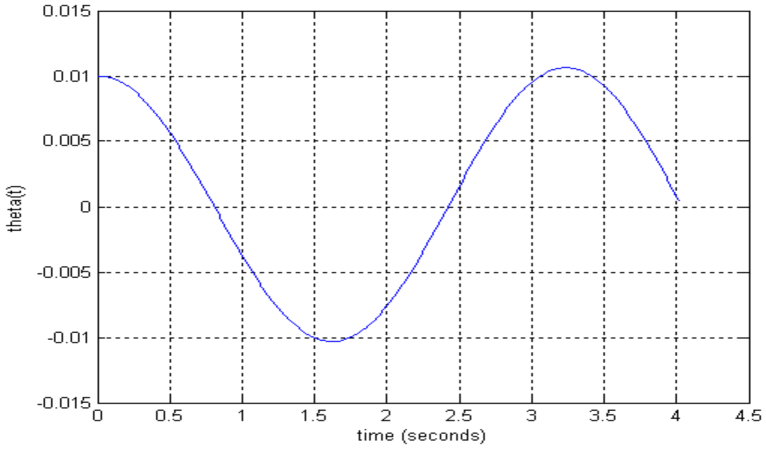


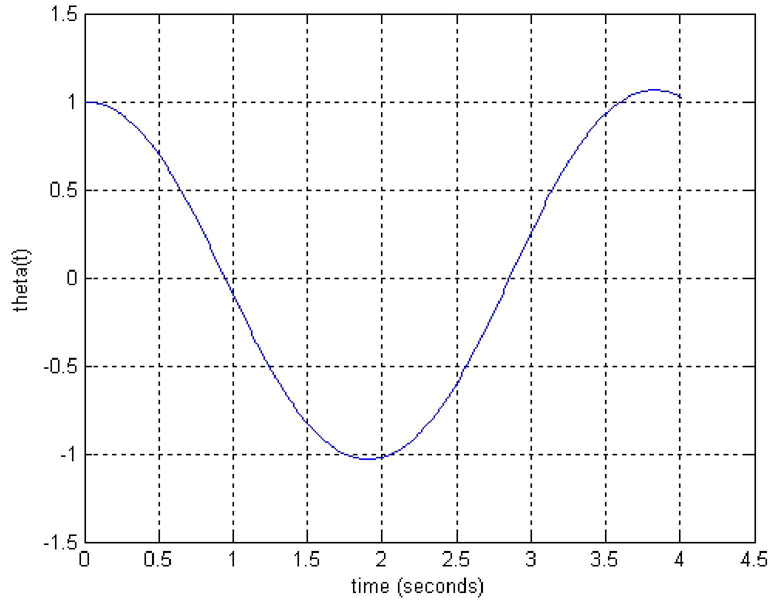
So, when the step size is very small the results are more accurate. But the running of the program took about more than 20 minutes to finish the calculations. Therefore it can use smaller h to shorten the time of execution.

Now we can investigate the small angle. For this investigation we will take the parameters of case c; **the time range t= 0 to T (period) (h=0.01**

f- Investigation of $\theta \approx \sin\theta$, θ_0=0.01

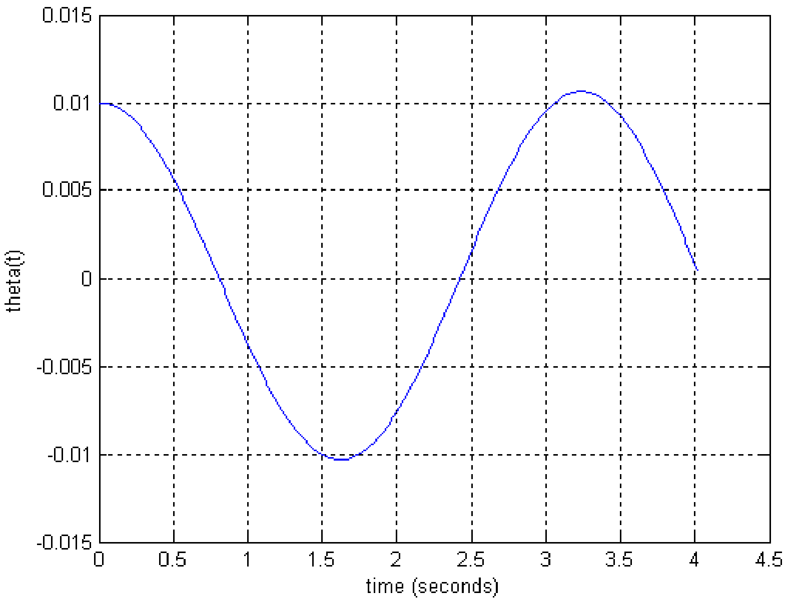
Case f is same as case c because in both the angle is very small.



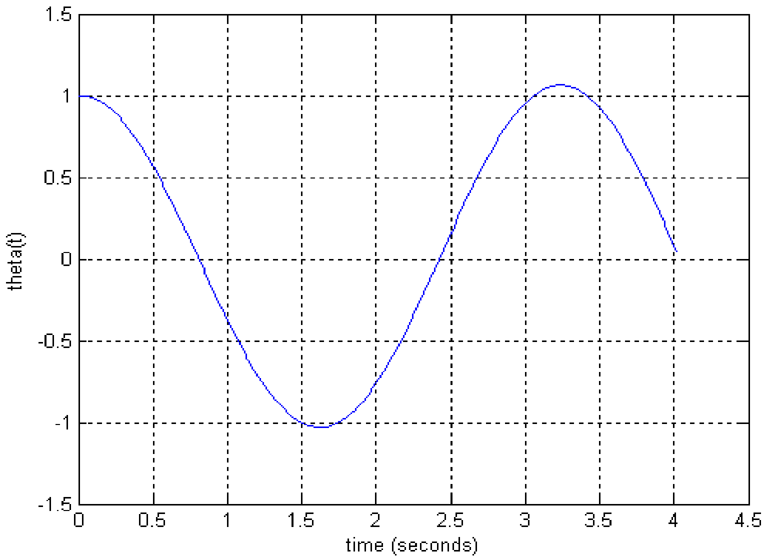


g- two cases with θ_0=1(we can see the differences at the points (0.5,0.5),(2,-1) and (3.5,1)) in both graphs below.

1- sinθ



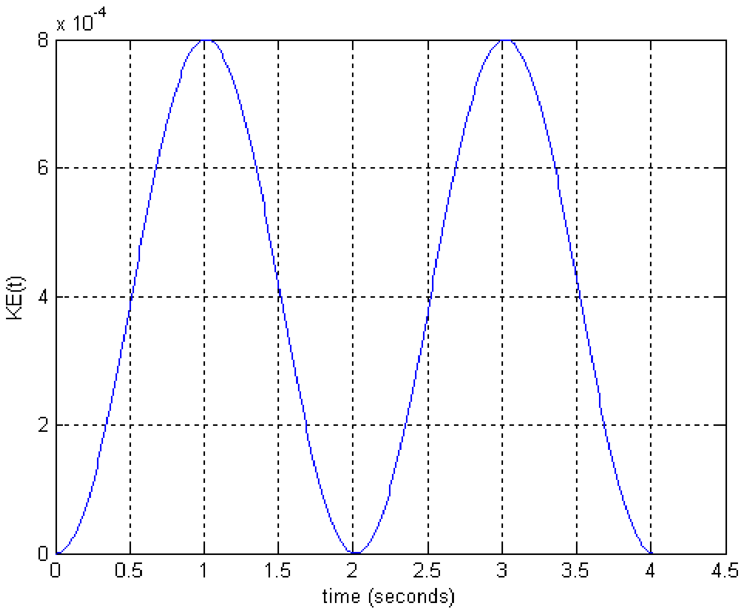
2- θ



the time is for one period. And it is clear that we take sinθ the graph shows one complete cycle while in the second graph, it is more than one cycle.

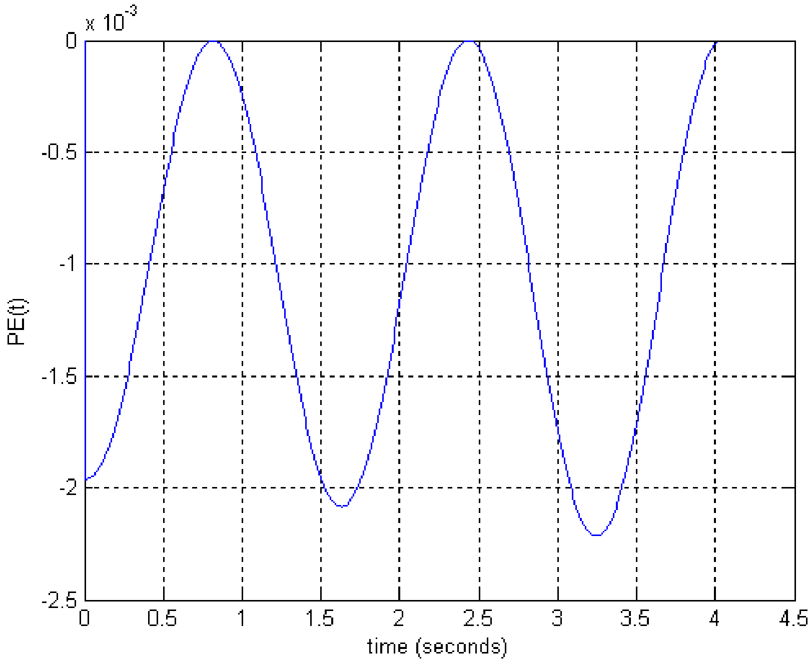
h)now we will investigate the kinetic energy for θ_0=0.01

h=0.01,



f) the potential energy for θ_0=0.01

h=0.01,



From the two above graphs, one can see that when the potential is at a maximum value the kinetic energy is at minimum value and vice versa.

5. the code

```
clear;clear memory;
%f = inline(`1./((x-0.3).^2 + 0.01) + 1./((x-0.9).^2 + 0.04)-6');
%f= inline('y*sin(x)+x*cos(y)','x','y')
%f(pi,2*pi)
%defining the constantsand boundary conditions
theta0=0.01;
w0=0;
l=4;ga=9.8;
wn=sqrt(ga./l);

ke=[0]
%domain of time
t0=0;
T=2.*pi/wn;
tf=T;
%step size h
h=0.01;
% defining the functions
g=inline('-9.8./l.*sin(wn.*t)','t','wn','l');
%g=inline('-9.8./l.*(wn.*t)','t','wn','l');

f=inline('w');
```



```
%counter
i=1;
%initializing the vectors of t, theta and angular velocity w
t=[t0];theta=[theta0];w=[w0];
pe=ga.*l.*(cos(wn.*t(i))-1);
pe=[pe];
while t(i)<tf
   kp1=h*g(theta(i),wn,l);
   kp2=h.*g(theta(i)+kp1./2,wn,l);
   kp3=h.*g(theta(i)+kp2./2,wn,l);
   kp4=h.*g(theta(i)+kp3,wn,l);
  w(i+1)=w(i)+(kp1+2.*kp2+2.*kp3+kp4)./6
 %------------------------
 k1=h.*f(w(i));
 k2=h.*f(w(i)+k1./2);
 k3=h.*f(w(i)+k2./2);
 k4=h.*f(w(i)+k3);
theta(i+1)=theta(i)+(k1+2.*k2+2.*k3+k4)./6
 %------------------------
 pe(i+1)=ga.*l.*(cos(wn.*t(i))-1);
 ke(i+1)=1/2.*l.^2.*theta0.^2.*(sin(wn.*t(i))).^2;
   i=i+1;
   t(i)=t(i-1)+h;

end
plot(t,theta)
grid on
set(get(gca,'XLabel'),'String','time (seconds)')
set(get(gca,'YLabel'),'String','theta(t)')
```

Conclusion

In this study, the non-linear motion of a simple pendulum was numerically investigated using the fourth-order Runge-Kutta (RK4) method. Through a comparative analysis, it was demonstrated that the RK4 algorithm provides a highly accurate approximation of the pendulum's equation, particularly when adopting smaller step sizes (h). While the small-angle approximation remains useful for simplified harmonic motion, numerical integration allows for a comprehensive understanding of pendulum dynamics beyond these limitations, consistent with the foundational principles of non-linear dynamics (Strogatz, 2014).

The investigation highlighted that the accuracy of the numerical solution is highly sensitive to the chosen step size, confirming that smaller intervals significantly reduce computational error, albeit at the cost of longer execution times. Furthermore, the energy analysis successfully verified the conservation of energy, illustrating the inverse relationship between kinetic and potential energy in a frictionless system. This work reinforces the efficacy of numerical methods as essential tools for solving ordinary differential equations in physical science and engineering (Butcher, 2016; Kreyszig, 2011). Future research could extend this model to include damping factors and external periodic forces to further analyze real-world oscillating systems.

Compliance with ethical standards

Disclosure of conflict of interest

The authors declare that they have no conflict of interest.

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